Zermelo's Theorem in Game Theory

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Abstract

We already know that tic-tac-toe is a solved game, meaning it is determined if both players play their best. This is, however, not so clear for the game of Chess. We will first start with a good game of tic-tac-toe, then refine the intuition into rigor on our way to proving the first formal theorem in the theory of games, credited to Ernst Zermelo. In full generality, we show that in two-player finite games of perfect information, **there is always a strategy** to reach a win or draw; with no regard to what the strategy should be.

One could then ask the following question. Given a winning position, how quickly can a win be forced? We conclude the talk with a reflection on Zermelo's findings as well as the powerful, non-intuitive consequences of his results.

OVERVIEW

Contents

1 A Game of Tic-Tac-Toe

1.1 Introduction

On the 19th of February, I started my talk by playing **this game** with a member of the audience.

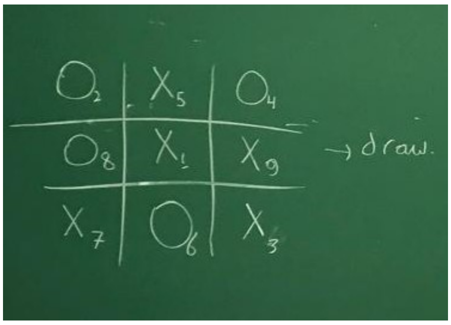
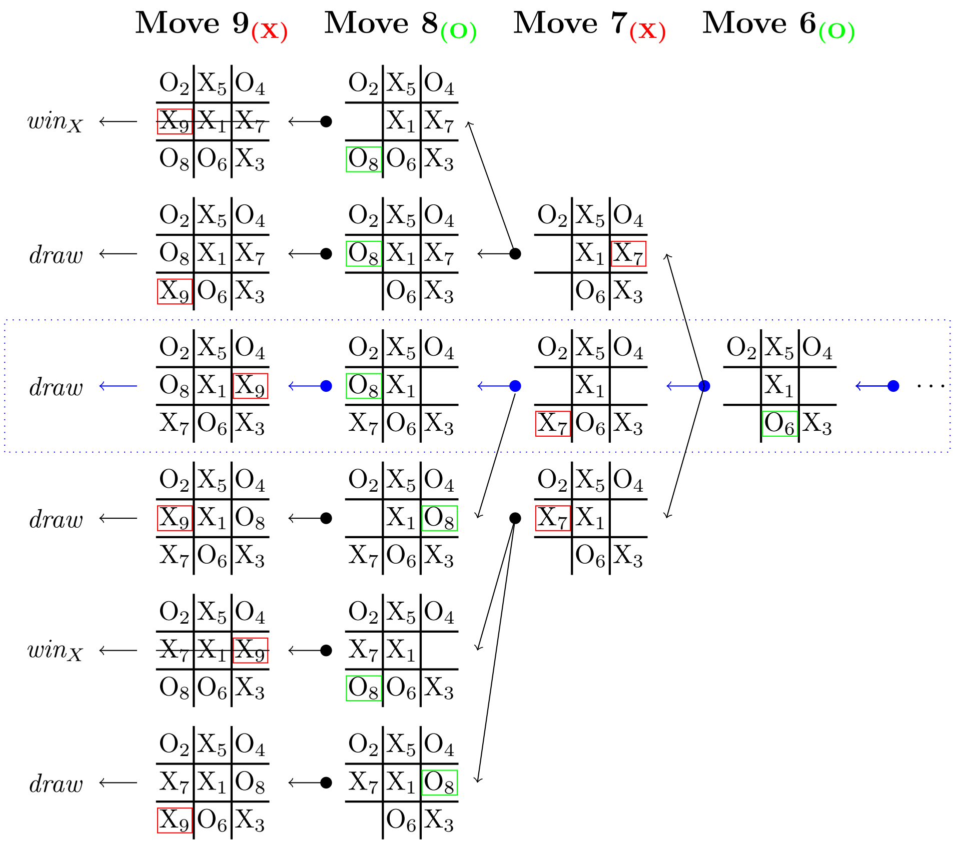


Figure 1: Game of Tic-Tac-Toe

Unsurprisingly, this game ended in a draw. As **O**-mar, I spent the preceding 2 hours perfecting my strategy to **force** this result.

A perfect game entails making the best moves at each phase of the game. As long as I played *perfectly,* I could force a draw.

My opponent, **X** was not too shabby either, who played a *perfect* game. Knowing the indices $\mathbf{X}_i$, $\mathbf{O}_i$ denote the progression of the game, let us study its anatomy.



We cannot list all the deviations of the game, as there are 9! [1] of those. Instead, let us consider the **sub-game** which starts at the **6**th **node,** and follows the **blue** path.

[1] 9! > 8.

I had to choose a square In **Move 6**. My **strategy** was to

1. assume **X** will play the best moves;
2. think about the **terminating positions** of the game.

Notice that any other **move** at the $\mathbf{6}^{th}$ **node** would have led to win_X, assuming *perfect play.*

1.2 Key Takeaways

Indeed, Tic-tac-toe is a solved game, and its solution is a draw assuming perfect play. Proof of this fact is neither provided in this paper, nor in Zermelo's argument, which is of the existence type. The fact is, however, fairly easy to see here.

With this brief introduction, we should be ready to sculpt our understanding into formal rigor.

2 The Theorem of Zermelo

We proceed with our discussion by formalising the simple notions we have covered so far, with a main goal of building up to Zermelo's theorem.

2.1 Formalisation

Definition 2.1 (Game)**.** A two player game with players $P_1,\ P_2$ consists of

1. a set of moves;
2. a set of rules that govern which positions are legal (allowed) in each turn;
3. a set of terminal positions where the game ends. Each terminal position has an associated outcome $\bar{\Omega} \in \{W_{P_1}, D, L_{P_1}\} =: \Omega$.

A **sub-game** is a subset of a game starting at move k less than the maximum game length.

This is a simple formalisation of the natural concept of a two-player game.

Definition 2.2 (Move)**.** A move is a sequence of actions in which the players alternate with each turn.

In a game of tic-tac-toe, we can denote the square in the $i-$th row and the $j-$th column with a_{ij}. Then, in the example we saw in 1.1,

$$m_6 = (a_{22}, a_{11}, a_{33}, a_{13}, a_{12}, a_{32})$$

denotes the sixth move that was played in the game. We omit the signs, as it is understood that **X** starts and **O** continues.

Exercise. Write down the full game moves.

Definition 2.3 (Perfect Information)**.** A game of perfect information is a game where at each decision node, each player knows where they are and the path that got them there.

The best way to illustrate this idea is by a non-example. A game of cards is **not** a game of perfect information, and the reasoning behind this fact is that the cards with each player are secret. It constitutes a game of **secret** information.

Definition 2.4 (Strategy)**.** A strategy is a complete plan of actions; it specifies which action will take place at each decision node.

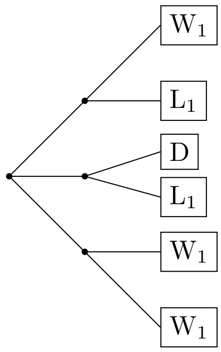
My strategy for the game in 1.1 was to start correctly. If a_{22} is not occupied, I would play it immediately. Otherwise I would focus on playing one of the corners $a_{11},\ a_{13},\ a_{31},\ a_{33}$. Then, the goal would be to not let **X** win; that is, prevent the game from terminating with win_X.

It is helpful to think of the game strategy as a map of directions, which factors out the outcome of each move, considers the opponent's possible moves, and decides the suitable way to proceed.

This is a good moment to stop and play a game. ;)

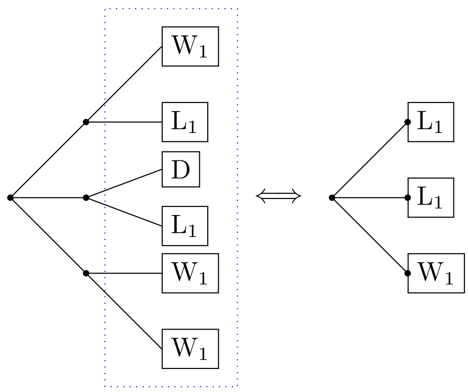
2.2 Dichotomy of the Game Tree

Example 2.5 (Game)**.** Let this be a two player game with players P_1, P_2.



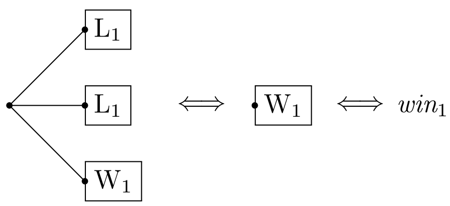
If the players take turns, who can force a win?

Solution. We start by considering the terminating sub-games of length 1. The game terminates at the second node, so P_2 will make the move. Assuming he plays perfectly, we see that P_2 makes the following choices:



1. in the first node, P_2 chooses L_1 over W_1, in order to win;
2. in the second node, P_2 chooses L_1 over D, in order to win;
3. in the third node, however, P_2 has no choice but W_1, so P_2 loses.

The first turn is now for P_1, for which we have



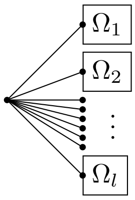
Therefore a win for P_1. □

This idea is the foundation of Zermelo's proof; that is, we can always reduce the game tree in the aforementioned construction.

Theorem 2.6 (Zermelo)**.** *In all finite two-person games of perfect information, either one player has a strategy to force a win, or both players have a strategy to force a draw.*

Remark. One could equivalently formulate the theorem as follows. For two players P_1, P_2 : either P_1 can force a win, or P_2 can force a win, or both players can force a draw.

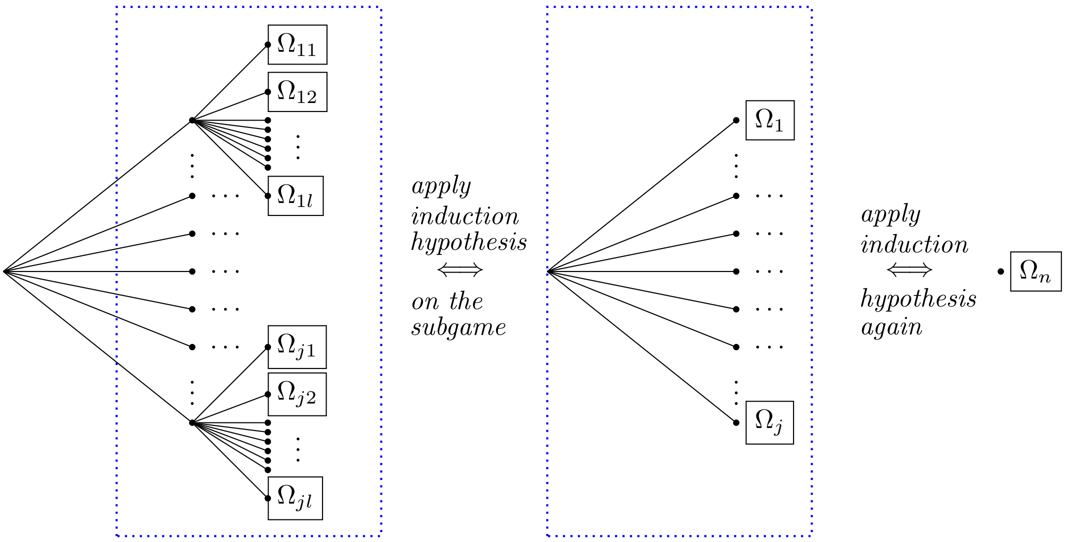
Proof. Induction on maximum length of the game. First, define $\Omega := \{W_1, L_1, D\}$ to be the set of outcomes. The outcome is pre-determined for $n = 0$, such that the outcome $\Omega_0 \in \Omega$. Next, we consider the case where $\boxed{n = 1}$. Only P_1 makes a move, and the game will be of the form



where the outcomes $\Omega_i \in \Omega$ are wins, losses, or draws. We divide this into cases.

1. If all $\Omega_i = W_1$, then clearly P_1 has a strategy to force a win.
2. If there exists Ω_k such that $\Omega_m = W_1$, then P_1 has a strategy to play Ω_k and force a win.
3. If all $\Omega_i \neq W_1$ but there exists an $\Omega_m = D$ then playing Ω_m is a strategy to a draw.
4. If all $\Omega_i = L_1$, then P_2 has a strategy to force a win, that is to do nothing.

Therefore the statement is immediately true for all games of length 1. Assume that the claim holds for all $k < n$. To conclude the argument, it is enough to invoke the hypothesis by reducing the game length to $n - 1$, which is immediate if we consider the terminating sub-games. And now consider the case of $k = n$, which for visual clarity we demonstrate a game of length $\boxed{n = 2}$.



Now that we have reduced the tree to length $n - 1$, we can apply the induction hypothesis such that the claim immediately follows! $\square$

3 Closing Words

3.1 Question by Kalmár

Building up upon the previous result, one could ask the following question. In a **winning position**, how quickly can a player force a win? Here is what Kalmár claims.

Corollary 3.1 (Kalmár, 1928)**.** *In less than n moves, that is the maximum length of the game.*

Proof. Assume the contrary, that the number of moves to win, n_w, is less than n, the length of the game. Then, this implies that the game has terminated before the winning position is reached.

Next, notice that there is an associated outcome with the termination of the game. Since this was a **winning position**, then there clearly must have been a winning strategy in less than n moves. This concludes the argument. $\square$

3.2 Reflection

We note, however, that Zermelo's proof shows the existence of a solution, but does not explicitly provide it. The statement is applicable to all finite games of perfect information, including Chess. The game of Chess is very complicated; it is however finite.

It is not so clear whether the game of Chess is a win, a loss, or a draw. For instance: Garry Kasparov, a chess Grandmaster, stated that chess is a draw. This came after a series of matches with Deep Blue, a computer engine which used the very same idea of backwards induction.

Others say that white can force a win by the "first-move" advantage. Statistically speaking, white often prevails when playing against black.

In any case, Zermelo has somewhat bad news for all chess fans out there, mainly due to the fact that their beloved game is determined. The only condolence for them is the fact that the solution is not yet known, and that no human mind can grasp the perfect strategy. But, with the constant surge of technology, and the consistent breakthroughs we see everyday, this reality might be changing very soon. $\boxed{\xi}$

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