Mathematical Modeling
Solution Sheet

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Abstract

This document outlines detailed solutions to homework assignments for Mathematical Modeling, taught by Professor Nikolai Leopold in the Spring of 2025. Written in an expository style, the goal of this document is to familiarise and equip the reader with the right ideas for a more developed treatment of the subject.

*The author wishes to acknowledge and thank Professor Nikolai Leopold for his valuable remarks and comments.

Contents

Sheet One

Keywords

Newton's Law of Cooling, Separation of Variables, Initial Conditions, Linear Inhomogeneous Differential Equations.

Contents

Exericse 1. *A cup of tea at* 90° *C is in a room at constant temperature of* 20° *C. By Newton's Law of Cooling, the change of the temperature in time is proportional to the difference between the current temperature of the tea and the room temperature. It is not affected by the amount of tea.*

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a) *Derive a differential equation that models the temperature* $T(t)$ *over time. Afterwards, find the solution of the differential equation.*

Proof. Let $k > 0$ and T be the temperature of tea at time t, with initial temperature T_0. The following differential equation

$$\frac{d}{dt}\,T = -k(T - 20)$$

expresses the assumption that the change in temperature is proportional to the difference of T and 20° C. By separation of variables, we obtain

$$\begin{aligned}\frac{dT}{T-20} = -kdt \implies & \int_{T_0}^{T} \frac{d\tilde{T}}{\tilde{T}-20} = \int_0^t -k\; d\tilde{t} \\ \implies & \ln(\tilde{T}-20)\Big|_{T_0}^{T} := \ln(T-20) - \ln(T_0 - 20) = -kt \\ \overset{e^{(\cdot)}}{\implies} & T - 20 = e^{-kt}(T_0 - 20) \implies T = 20 + e^{-kt}(T_0 - 20)\end{aligned}$$

For an initial temperature of 90° C, we get that $T(t) = 20 + e^{-kt} \cdot 70$ solves the differential equation. $\boxed{\xi}$

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b) *The temperature of the tea is* 70° *C after* 5 *minutes. Determine the constant which describes the speed of cooling. When will the temperature of the tea be* 40° *C?*

Solution. It is given that $T(5) = 70$, so that

$$70 = T(5) := 20 + 70e^{-k\cdot 5} \implies e^{-k\cdot 5} = \frac{5}{7} \implies -5k = \ln(\frac{5}{7}) \implies k = -\frac{\ln(\frac{5}{7})}{5}$$

gives the cooling rate. For $T(t) = 40$, this is just

$$\begin{aligned}40 = T(t_{40}) = 20 + 70e^{-k\cdot t_{40}} \implies & \frac{20}{70} = e^{-k\cdot t_{40}} \\ \implies & \ln(\frac{2}{7}) = -k \cdot t_{40} \\ \implies & -\frac{\ln(\frac{2}{7})}{k} := \cancel{-}\frac{\ln(\frac{2}{7})}{\cancel{-}\frac{\ln(\frac{5}{7})}{5}} = t_{40} \\ \implies & t_{40} = 5 \cdot \frac{\ln(\frac{2}{7})}{\ln(\frac{5}{7})} \approx 18.62 \text{ minutes.}\end{aligned}$$

$\boxed{\xi}$

Exercise 2. *Find the solution of the differential equation* $\frac{d}{dx}y(x) = 2xy(x) + x^3$.

Solution. Note that the aforementioned differential equation is one that is *linear* and *inhomogeneous*. We therefore make use of the Ansatz $y = uv$ to get

$$u'v + uv' =: \frac{d}{dx}\underbrace{(uv)}_{:=y} = 2x \cdot uv + x^3. \tag{*}$$

Next, notice that the choices

$$u' = 2x \cdot u \qquad \& \qquad v' = \frac{x^3}{u}, \tag{i.1}$$

satisfy (*) by design. We equated the two sides of (*) by comparison. This is the core idea, and with that we may proceed to solve two simpler differential equations, starting with u.

$$u' = 2x \cdot u \implies \int_{u_0}^{u} \frac{du}{u} = \int_{x_0}^{x} 2x \; dx \implies \underbrace{\ln(u) - \ln x_0}_{\ln(\frac{u}{u_0})} = x^2 - x_0^2$$
$$\implies u = u_0 e^{x^2 - x_0}$$

Then, v' may be written as

$$\frac{d}{dx}v = u_0^{-1}x^3 e^{(x_0^2 - x^2)} \implies v = v_0 + \underbrace{u_0^{-1}e^{x_0^2}}_{\textbf{constants}} \cdot \int_{x_0}^{x} x^3 e^{-x^2} dx$$

with the integral evaluating to

$$\int_{x_0}^{x} x^3 e^{-x^2} dx \underset{dt=2xdx}{\overset{t=x^2}{=}} \frac{1}{2}\int_{x_0^2}^{x^2} te^{-t}dt = \frac{1}{2}\left(-te^{-t} - \overbrace{\int -e^{-t} \; dt}^{:=e^{-t}}\right)\Bigg|_{x_0^2}^{x^2} = -\frac{1}{2}e^{-t}(t+1)\Big|_{x_0^2}^{x^2}$$
$$= \frac{1}{2}\left(e^{-x_0^2}(x_0^2+1) - e^{-x^2}(x^2+1)\right).$$

Therefore we get

$$v = v_0 + \frac{1}{2}u_0^{-1}e^{x_0^2}\left(\cancelto{1}{e^{-x_0^2}}(x_0^2+1) - e^{-x^2}(x^2+1)\right)$$
$$= v_0 + \frac{1}{2}u_0^{-1}\left(x_0^2 + 1 - e^{x_0^2 - x^2}(x^2+1)\right)$$

Finally, recall that $y = uv$ gives the solution.

$$y = u_0 \cdot e^{x^2 - x_0^2}\left(v_0 + \frac{1}{2}u_0^{-1}\left(x_0^2 + 1 - e^{x_0^2 - x^2}(x^2+1)\right)\right)$$
$$= \underbrace{u_0 v_0}_{:=y_0} e^{x^2 - x_0^2} + \frac{1}{2}e^{x^2 - x_0}(x_0^2 + 1) - \frac{1}{2}(x^2+1).$$

$\boxed{\xi}$

Remark. For more on this technique, seek page 5 of lectures 2, 3.

Exercise 3. *Find the solution of the differential equation* $\frac{d}{dx}y(x) = 2xy(x) + (1+x^2)y^2(x)$.

Proof. Using the *ansatz* $y = uv$ we get that

$$\frac{d}{dx}\, uv = u'v + uv' = 2x \cdot uv + (1+x^2) \cdot u^2v^2$$

By comparison of terms, set

$$u' = 2xu \tag{*}$$
$$v' = (1+x^2) \cdot uv^2 \tag{**}$$

then clearly $u = C_u \cdot e^{x^2}$. The second equation gives

$$\begin{aligned} \frac{dv}{dx} =: v' = (1+x^2) \cdot C_u \cdot e^{x^2} \cdot v^2 &\implies \frac{1}{v^2}\, dv = C_u \cdot \left(e^{x^2} + x^2 e^{x^2}\right) \cdot dx \\ &\implies -\frac{1}{v} = C_u \cdot \int (1+x^2) \cdot e^{x^2} dx \\ &\implies v = -C_u^{-1} \cdot \frac{1}{\int (1+x^2) \cdot e^{x^2} dx}. \end{aligned}$$

All-in-all, this gives

$$y = \cancel{-C_u^{-1}} \cdot \frac{\cancel{C_u} \cdot e^{x^2}}{\int (1+x^2) \cdot e^{x^2} dx} = -\frac{e^{x^2}}{\int (1+x^2) \cdot e^{x^2} dx}.$$

We do not attempt to compute the integral $\int (1+x^2) \cdot e^{x^2} dx$, since it is non-elementary. This concludes the argument. $\boxed{\xi}$

Sheet Two

Keywords

Linear Homogeneous Differential Equations, Linear Inhomogeneous Differential Equations, the Logistic Equation, Vector Fields, Systems of Differential Equations.

Contents

Exercise 1 *Consider the second-order inhomogeneous differential equation*

$$\frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x) = e^x.$$

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a) *Find the general solution to the corresponding homogeneous equation.*

Solution. A general solution to the homogeneous equation of the form

$$a \cdot \frac{d^2}{d^2x}y(x) + b \cdot \frac{d}{dx}y(x) + c \cdot y(x) = 0$$

is well-studied*, and the idea is to consider the choice of $y = e^{\lambda x}$. This is great, since this choice

$$\begin{cases} y(x) = e^{\lambda x} \\ \frac{d}{dx}y(x) = \lambda e^{\lambda x} \\ \frac{d^2}{d^2x}y(x) = \lambda^2 e^{\lambda x} \end{cases} \implies a \cdot \lambda^2 e^{\lambda x} + b \cdot \lambda e^{\lambda x} + c \cdot e^{\lambda x} = e^{\lambda x}\left(a \cdot \lambda^2 + b \cdot \lambda + c\right) = 0$$

$$\implies a \cdot \lambda^2 + b \cdot \lambda + c = 0 \qquad \text{(since } e^{\lambda x} > 0\text{)}$$

in turn yields a polynomial equation, which we are very happy to solve. It simplifies the task to finding λ. Let us indeed proceed in that very-same spirit, and solve for $a = 1, b = -2, c = 2$.

$$1 \cdot \lambda^2 - 3 \cdot \lambda + 2 = 0 \implies \lambda_1 = 1, \lambda_2 = 2 \implies \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}.$$

Notice that we ended up with two solutions, when we were looking for one. This is a good moment to recall that any linear combination of homogeneous solutions gives a homogeneous solution,

$$\boxed{y(x) = Ae^x + Be^{2x} \qquad \text{for } A, B \in \mathbb{R}}$$

by linearity of the differential operator. As an exercise, try to verify yourself that

$$\frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x) = 0$$

for the choices of

- $y(x) = e^x$
- $y(x) = Ae^x$
- $y(x) = e^x + e^{2x}$
- $y(x) = Ae^x + Be^{2x}$.

$\boxed{\xi}$

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*see page 10, lectures 2 and 3 on moodle.

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b) *Consider the second-order inhomogeneous differential equation*

$$\frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x) = e^x.$$

Find the general solution to the inhomogeneous equation.

Proof. We will use the following lemma from class,

Lemma 1.5 (Lectures 2,3 - Page 9)**.** *Consider the following inhomogeneous equation*

$$a \cdot \frac{d^2}{d^2x}y(x) + b \cdot \frac{d}{dx}y(x) + c \cdot y(x) = f(x).$$

The general solution $y(x)$ may be written as

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the general solution of the homogeneous equation, and $y_p(x)$ is a particular solution of the inhomogeneuos equation.

In the previous problem, we established the homogeneous solution to be

$$y_h(x) = Ae^x + Be^{2x} \qquad \text{for } A, B \in \mathbb{R}.$$

If we can find a particular solution $y_p(x)$, then we are done. Now, to find $y_p(x)$, one may proceed in the spirit of *the quick method*[†] of undetermined coefficients. One attempts to guess an *Ansatz* of a similar structure to $f(x)$, equal to e^x in our case. Another way to proceed is with the *variation of constants*, which is more informative.[‡]

The idea is to equate $y_p(x)$ to $y_h(x)$, but with coefficients that vary in x. Instead of constants A, B, we write

$$y_p(x) = A(x)e^x + B(x)e^{2x}$$

as functions of x. This reduces the problem to that of finding the coefficients $A(x), B(x)$, since that automatically describes $y_p(x)$. With this description, let us compute $\frac{d}{dx}y_p(x), \frac{d^2}{d^2x}y_p(x)$.

$$\frac{d}{dx}y_p(x) := A'(x)e^x + B'(x)e^{2x} + A(x)e^x + 2 \cdot B(x)e^{2x}.$$

Let us enforce a restriction on $A'(x), B'(x)$. This will become very useful in a moment.

$$A'(x)e^x + B'(x)e^{2x} = 0. \tag{1}$$

[†]Page 2, Lectures 4,5, on moodle.

[‡]This is a subjective opinion.

This implies

$$\frac{d}{dx}y_p(x) =^{(1)} \underbrace{A'(x)e^x + B'(x)e^{2x}}_{=\ 0 \text{ by assumption (1)}} + A(x)e^x + 2\cdot B(x)e^{2x} = A(x)e^x + 2\cdot Be^{2x}. \tag{*}$$

Next, we compute $\dfrac{d^2}{d^2x}y_p(x)$.

$$\begin{aligned}\frac{d^2}{d^2x}y_p(x) &= \frac{d}{dx}\left(\frac{d}{dx}y_p(x)\right)\\ &=^{(*)} \frac{d}{dx}\left(A(x)e^x + 2\cdot B(x)e^{2x}\right)\\ &= A'(x)e^x + 2\cdot B'(x)e^{2x} + A(x)e^x + 4\cdot B(x)e^{2x} \end{aligned}\tag{**}$$

In total, we obtain

$$\begin{cases} y(x) = A(x)e^x + B(x)e^{2x} \\ \dfrac{d}{dx}y(x) = A(x)e^x + 2\cdot Be^{2x} & \text{(by *)} \\ \dfrac{d^2}{d^2x}y(x) = A'(x)e^x + 2\cdot B'(x)e^{2x} + A(x)e^x + 4\cdot B(x)e^{2x}. & \text{(by **)} \end{cases}$$

Plugging this into the inhomogeneous differential equation gives

$$\begin{aligned} e^x &= \frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x)\\ &= \overbrace{A'(x)e^x + 2\cdot B'(x)e^{2x} + A(x)e^x + 4\cdot B(x)e^{2x}}^{=\frac{d^2}{d^2x}y(x)} - 3\overbrace{\left(A(x)e^x + 2\cdot B(x)e^{2x}\right)}^{=\frac{d}{dx}y(x)} + 2\overbrace{\left(A(x)e^x + B(x)e^{2x}\right)}^{=y(x)}\\ &= A'(x)e^x + 2\cdot B'(x)e^{2x} + A(x)e^x\underbrace{(1-3+2)}_{=0} + B(x)e^{2x}\underbrace{(4-3\cdot 2+2)}_{=0}\\ &= A'(x)e^x + 2\cdot B'(x)e^{2x}. \end{aligned}\tag{2}$$

Now that we have two equations (1), (2) in the two variables $A'(x), B'(x)$, we may express it as a system of linear equations,

$$\begin{cases} A'(x)e^x + B'(x)e^{2x} = 0 & (1)\\ A'(x)e^x + 2\cdot B'(x)e^{2x} = e^x & (2)\end{cases} \implies \begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x}\end{pmatrix}\begin{pmatrix} A'(x) \\ B'(x)\end{pmatrix} = \begin{pmatrix} 0 \\ e^x \end{pmatrix}.$$

which has the solution vector $\binom{A'(x)}{B'(x)} = \binom{-1}{e^{-x}}$. To find $A(x), B(x)$, we integrate disregarding the constants [§] to get

$$A'(x) = -1 \implies A(x) = -x$$
$$B'(x) = e^{-x} \implies B(x) = -e^{-x}.$$

That in turn yields $y_p(x) = \overbrace{-xe^x - e^x}^{:=A(x)e^x + B(x)e^{2x}}$. Utilising the homogeneous solution $y_h(x)$ from the previous problem, (1.5) allows us to write

$$\begin{aligned} y(x) &= \overbrace{Ae^x + Be^{2x}}^{y_h(x)} + \overbrace{-xe^x - e^x}^{y_p(x)} \\ &= \boxed{Ae^x + Be^{2x} - xe^x \quad \text{for } A, B \in \mathbb{R}.} \end{aligned}$$

$\boxed{\xi}$

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c) *Consider the second-order inhomogeneous differential equation*

$$\frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x) = e^x.$$

Find the specific solution that satisfies the conditions $y(0) = 1$ *and* $(\frac{d}{dx}y)(0) = 0$.

Solution. Simply, one combines the result from the previous exercise

$$y(x) = Ae^x + Be^{2x} - xe^x$$

with the initial conditions to obtain a system of equations

$$\begin{aligned} 1 = y(0) &= A + B \\ 0 = y'(0) &= Ae^0 + 2Be^{2\cdot 0} - e^0 - 0 \cdot e^0 \\ &= A + 2B - 1 \end{aligned}$$

which has the solutions $A = 1,\ B = 0$. We write

$$\boxed{y(x) = e^x - xe^x.}$$

$\boxed{\xi}$

[§] We can do this, since the constants are accounted for in the homogeneous equation. For example, if $A(x) = -x + c$ then $A(x)e^x = -xe^x + ce^x$ which we combine with the constant term Ae^x from $y_h(x) = Ae^x + Be^{2x}$.

Exercise 2 *The logistic equation is a model for population growth with a maximum sustainable population. It is given by*

$$\frac{d}{dt}P(t) = rP(t)\left(1 - \frac{P(t)}{K}\right),$$

where $P(t)$ *denotes the size of the population at time* t, $r > 0$ *is the (constant) growth rate of the population, and* $K > 0$ *is the maximum sustainable population.*

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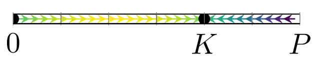
a) *Sketch the direction field for the logistic equation.*

Sketch. An ordinary differential equation $\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t))$ is defined by a vector field f. In this case, the equation only has one variable, and thus the vector field is one-dimensional.[¶] Moreover, f is given exactly by the logistic equation

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \;\; P \mapsto f(P) = rP(1 - \frac{P}{K}).$$

A very rough sketch of the vector field is therefore

Vector Field for Logistic Equation $f(P) = rP\left(1 - \frac{P}{K}\right)$



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b) *Explain in words how* $P(t)$ *changes when* $P(t) \ll K$, $P(t) = K$, *and* $P(t) > K$. *How does* $P(t)$ *behave for large times?*

Explanation. There are three cases to consider.

$P(t) \ll K$ | First, let it be clear that $P(t) \ll K$ means that $P(t)$ is significantly smaller than K. If this is the case, then the fraction $\frac{P(t)}{k}$ is very small, and may be neglected. Then, $1 - \frac{P(t)}{K}$ is close to 1, and

$$\frac{d}{dt}P(t) \approx rP(t).$$

This is the equation for exponential growth with rate r. This means that $P(t)$ grows *almost* exponentially *towards* K, and we may write $P(t) \approx e^{rt}$.

$P(t) = K$ | In that case, $1 - \frac{P(t)}{K} = 0$, and

$$\frac{d}{dt}P(t) = rP(t)\cancelto{0}{(1 - \frac{P(t)}{K})} = 0$$

[¶]Seek page 8 of Lectures 4,5 for examples.

implies that $P(t)$ is a constant, fixed value. This is a natural consequence, since the constant K does not depend on t. The population is then in a state of equilibrium at $P(t) = K$.

$P(t) > K$ | Finally, $1 - \frac{P(t)}{K} < 0$ gives

$$\frac{d}{dt}P(t) = \underbrace{rP(t)(1 - \frac{P(t)}{K})}_{<0} < 0$$

meaning that the population is too big, and decays *towards* $P(t) = K$.

In summary, the relation between $P(t)$ and K characterizes $\frac{d}{dt}P(t)$. Precisely: as t gets larger, the population $P(t)$ tends to a state of equilibrium $P(t) = K$. $\boxed{\xi}$

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c) *Find the solution to the logistic equation with initial condition $P(0) = P_0$.*
Solution. First, let us re-write the equation as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

Next, separate the variables and rewrite

$$\frac{dP}{P \cdot (1 - \frac{P}{K})} = r\ dt \qquad \text{as} \qquad \frac{K}{P \cdot (K - P)} \cdot dP = r\ dt.$$

Partial Fractions. The left-hand side is difficult to integrate in this form. It would be much easier if we could write it as two terms $\frac{A}{P}$ and $\frac{B}{K-P}$ for some constants A, B. Luckily, the partial fraction method provides just that. Assume indeed that

$$\frac{K}{P \cdot (K - P)} = \frac{A}{P} + \frac{B}{K - P} \overset{\times P(K-P)}{\Longrightarrow} K = A \cdot (K - P) + B \cdot P \implies K = P \cdot (B - A) + A \cdot K.$$

Notice that the first statement in this chain of implications is an identity[‖] on P, and thus we may plug-in $P = 0$ in the final statement to get

$$P \cdot (B - A) + A \cdot K = K \overset{P=0}{\Longrightarrow} A = 1.$$

By substituting $A = 1$ into the equation and solving for B, we get that

$$P \cdot (B - 1) + K = K \implies B = 1$$

As an exercise, you may check that $\frac{K}{P \cdot (K - P)} = \frac{1}{P} + \frac{1}{K - P}$.

[‖]is true for all P.

Integration. With the partial fractions expression, we write

$$\frac{K}{P\cdot(K-P)}\cdot dP = r\ dt \quad \overset{\text{Partial Fractions}}{\Longrightarrow} \quad \frac{1}{P}+\frac{1}{K-P} = r\ dt \Longrightarrow \int \frac{1}{P}dp + \int \frac{1}{K-P}\ dp = \int r\ dt$$

$$\Longrightarrow \ln(P)-\ln(K-P) = rt + C \Longrightarrow \ln\left(\frac{P}{K-P}\right) = rt + C.$$

$\mathbf{P(t)=?}$ Solving for P, we get

$$\ln\left(\frac{P}{K-P}\right) = rt + C \Longrightarrow \frac{P}{K-P} = e^{rt}\cdot e^{C} \Longrightarrow P(t) = \frac{Ke^{rt}\cdot e^{c}}{1+e^{rt}\cdot e^{C}} \quad \text{for} \quad C\in\mathbb{R}.$$

Initial Condition. To find the constant C (rather e^C), we utilise the initial condition $P(0)=P_0$ to write

$$P_0 = P(0) = \frac{Ke^{r\cdot 0}\cdot e^{C}}{1+e^{r\cdot 0}\cdot e^{C}} = \frac{Ke^{C}}{1+e^{C}} \Longrightarrow e^{C} = \frac{P_0}{K-P_0}.$$

Plug this into the expression to get

$$P(t) = \frac{Ke^{rt}\cdot A}{1+e^{rt}\cdot A} \quad \text{for } A = \frac{P_0}{K-P_0}. \qquad \boxed{\xi}$$

Exercise 3 *Consider the system of ODEs*

$$\begin{cases} \frac{d}{dt}x(t) = x(t) - y(t) \\ \frac{d}{dt}y(t) = x(t) + y(t). \end{cases}$$

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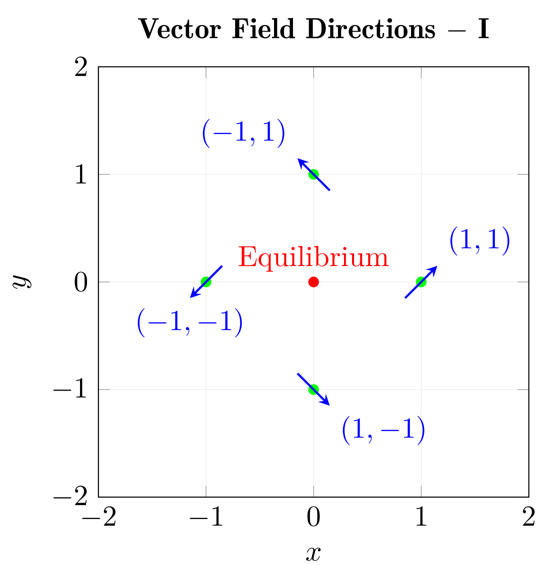
a) *Sketch the vector field.*

Sketch. Let us start with some observations, and compute gradient at different vectors $\begin{bmatrix} x \\ y \end{bmatrix}$.

1. $(x, y) = (1, 0) \implies \begin{cases} \frac{dx}{dt} = 1 - 0 = 1 \\ \frac{dy}{dt} = 1 + 0 = 1 \end{cases}$
2. $(x, y) = (0, 1) \implies \begin{cases} \frac{dx}{dt} = 0 - 1 = -1 \\ \frac{dy}{dt} = 0 + 1 = 1 \end{cases}$
3. $(x, y) = (-1, 0) \implies \begin{cases} \frac{dx}{dt} = -1 - 0 = -1 \\ \frac{dy}{dt} = -1 + 0 = -1 \end{cases}$
4. $(x, y) = (0, -1) \implies \begin{cases} \frac{dx}{dt} = 0 - -1 = 1 \\ \frac{dy}{dt} = 0 + -1 = -1 \end{cases}$

A positive gradient in x indicates growth in the $x-$direction, and a negative gradient in y indicates decay in the $y-$direction. This is a good point to stop and observe some nice drawings.



It seems that there is a tendency to go counter-clockwise. It is not clear just yet whether the trajectories converges inward or diverges outward. For this, let us compute the gradient for vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with greater magnitude.

1. $(x, y) = (2, 0) \implies \begin{cases} \dfrac{dx}{dt} = 2 - 0 = 2 \\ \dfrac{dy}{dt} = 2 + 0 = 2 \end{cases}$

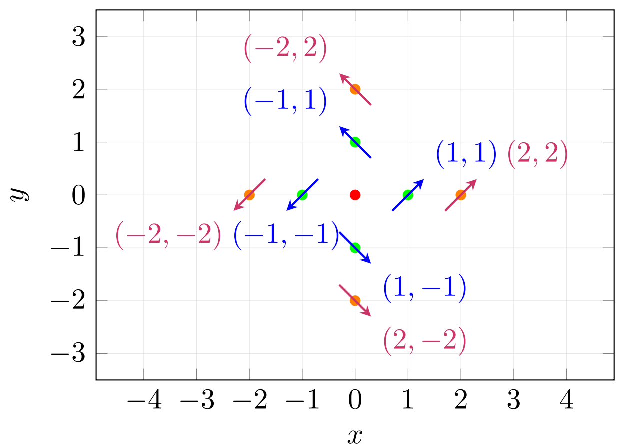
2. $(x, y) = (0, 2) \implies \begin{cases} \dfrac{dx}{dt} = 0 - 2 = -2 \\ \dfrac{dy}{dt} = 0 + 2 = 2 \end{cases}$

3. $(x, y) = (-2, 0) \implies \begin{cases} \dfrac{dx}{dt} = -2 - 0 = -2 \\ \dfrac{dy}{dt} = -2 + 0 = -2 \end{cases}$

4. $(x, y) = (0, -2) \implies \begin{cases} \dfrac{dx}{dt} = 0 - -2 = 2 \\ \dfrac{dy}{dt} = 0 + -2 = -2 \end{cases}$

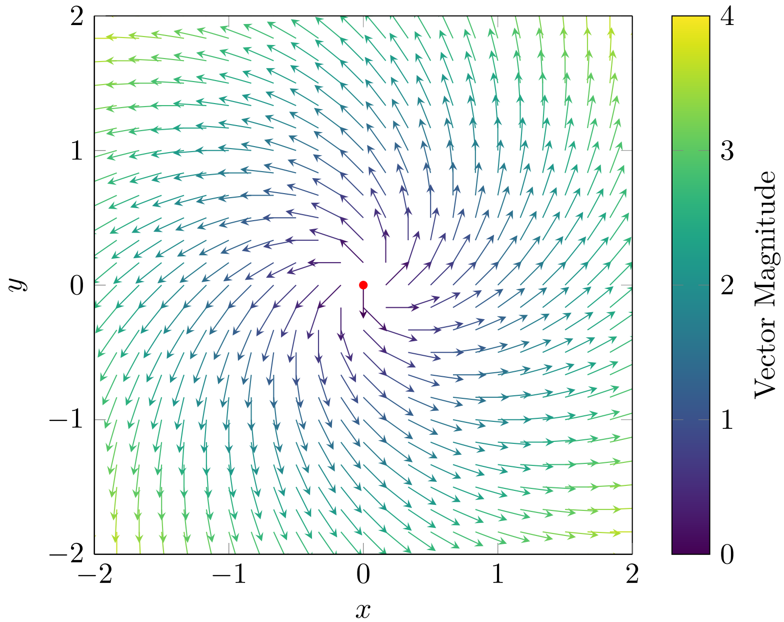
The gradients seem to get greater in magnitude. This indicates an *unstable* vector field whose trajectories diverge outwards with time.

Vector Field Directions − II



With these observations, the vector field should take on the form

Vector Field for $\frac{d}{dt}\binom{x}{y} = \binom{x-y}{x+y}$



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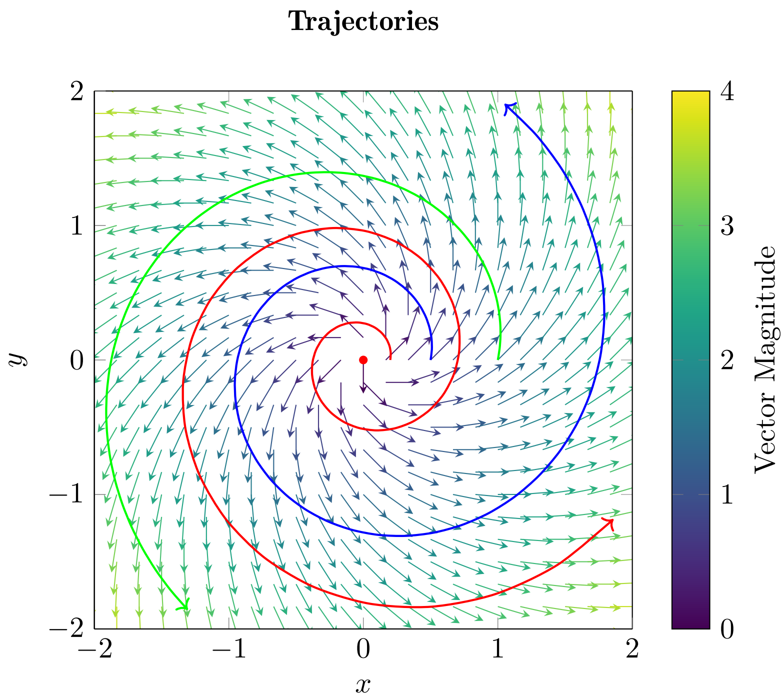
b) *Consider the system of ODEs*

$$\begin{cases} \frac{d}{dt}x(t) = x(t) - y(t) \\ \frac{d}{dt}y(t) = x(t) + y(t). \end{cases}$$

Using the vector field sketch, sketch a few representative solution trajectories in the phase space.

Sketch. Using our vector field sketch, we pick some starting points and see where the vector field flows them to.



ξ

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c) *Consider the system of ODEs*

$$\begin{cases} \frac{d}{dt}x(t) = x(t) - y(t) \\ \frac{d}{dt}y(t) = x(t) + y(t). \end{cases}$$

Determine if the orbits are periodic.
Hint: Look at the arrows in the vector field as you move away from the origin. Do they drive you further away, or do they guide you back toward a loop? If the arrows push you away, the orbits are not periodic.

Solution. Clearly, and as demonstrated above, the orbits are not periodic. ξ

Sketching a Vector Field, but Analytically – Bonus *Consider the system of ODEs*

$$\begin{cases} \frac{d}{dt}x(t) = x(t) - y(t) \\ \frac{d}{dt}y(t) = x(t) + y(t). \end{cases}$$

Describe the analytical behaviour of this system.

Description. First, observe that the system of equations is linear, which is very nice. Let us combine the two first-order differential equations into one degree-two differential equation.

$$x' = x - y \overset{\frac{d}{dt}}{\Longrightarrow} x'' = \overbrace{x'}^{:=x-y} - \underbrace{y'}_{:=x+y} = -2 \cdot \underbrace{y}_{:=x-x'} = -2(x - x') \implies x'' + 2x' - 2x = 0$$

This is exactly the same setup as in **1a)**. Proceed with the choice of $x = e^{\lambda t}$, and let us solve for λ.

$$x'' + 2x' - 2x = 0 \overset{x(t)=e^{\lambda t}}{\Longrightarrow} e^t(\lambda^2 + 2\lambda - 2) = 0 \implies \lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} = 1 \pm i.$$

Next, the following remark is quite useful,

Remark. If $\lambda_{1,2} = \alpha + i\beta$ are two complex solutions to the characteristic equation, then

$$e^{\alpha t}\cos(\beta t) \qquad \& \qquad e^{\alpha t}\sin(\beta t)$$

are two linearly independent solutions.**

The general solution $x(t)$ is the linear combination of all independent solutions. The remark thus allows us to write the solutions for $\alpha = 1, \beta = 1$ to get

$$x(t) = e^t(A\cos t + B\sin t).$$

Next, $y = x - x'$ gives

$$y(t) = \cancel{e^t(A\cos t + B\sin t)} - \Big(\cancel{e^t(A\cos t + B\sin t)} + e^t(-A\sin t + B\cos t)\Big) = e^t(A\sin t - B\cos t).$$

With the solution

$$\begin{cases} x(t) = e^t(A\cos t + B\sin t) \\ y(t) = e^t(A\cos t - B\sin t) \end{cases}$$

in hand, let us attempt to study the behaviour of $x(t), y(t)$ with time. First, the solution is unstable in the sense that its magnitude grows exponentially with time; credited to the factor e^t. The $\cos t, \sin t$ factors add a counter-clockwise rotation to the field. To conclude, the trajectories spiral outwards in a counter−clockwise direction. The vector $\vec{0}$ must therefore be the only equilibrium point. $\boxed{\xi}$

**We know this from class. See the Remark on Page 14, Lectures 2 and 3 on moodle.

Sheet Three

Keywords

Phase Space Trajectory, Vector Field, Conservation Law of Energy, Harmonic Oscillator, Global and Local Lipschitz Continuity.

Contents

Exercise 1 *Let the velocity be defined as* $v(t) = \frac{d}{dt}x(t)$.

---

a) *Express the second-order differential equation* $\frac{d^2}{d^2t}x(t) + \frac{k}{m}x(t) = 0$, *which models an undamped harmonic oscillator without external forcing, as an equivalent system of first-order ordinary differential equations using the velocity variable* $v(t)$.

Solution. If $v = x'$ as the question assumes, then $v' = x''$ and we may re-write the second-order differential equation as a system of two equations, both of first-order. $\boxed{\xi}$

$$x' = v \tag{iii.1}$$

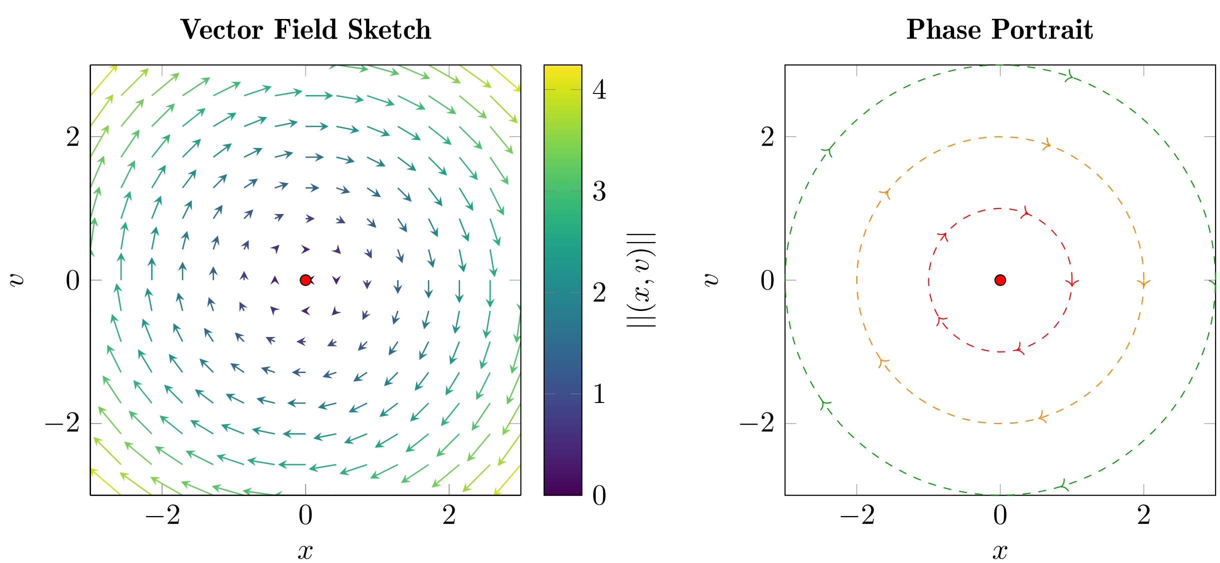
$$v' = -\frac{k}{m} \cdot x \tag{iii.2}$$

---

b) *Sketch the vector field and phase portrait corresponding to the system of first-order ODEs from part a) for the parameter values* $k = 2$ *and* $m = 2$.

Remark. The choice $k = m = 2$ gives rise to $\begin{cases} x' = v \\ v' = -x \end{cases}$. This is precisely **Example 1** from the lecture, differing only by a minus sign.[*] This difference is, in fact, reflected in the direction of rotation, which is clockwise compared to anti-clockwise sketch of the example. There, a similar phase portrait is offered as well. We proceed nevertheless without this knowledge.

Sketch. The system corresponding to the choices $k = m = 2$ is given by $f(x, v) = \binom{v}{-x}$. To plot its associated vector field and phase portrait, one simply computes the gradient at a few points in the $xv-$plane to get the vector field, then traces some trajectories along these gradient vectors. [†]



$\boxed{\xi}$

[*] See Page 8, Section 1.3, Lectures 4 & 5
[†] See (**ii.3**) for more on this technique.

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c) *Sketch the vector field and phase portrait for the same system with parameter values $k = 8$ and $m = 2$. Describe how and why the phase portrait changes when the parameter k is increased.*

Sketch. The system of equations $\begin{cases} x' &= v \\ v' &= -\frac{k}{m} \cdot x \end{cases}$ tells us that $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = -\frac{k}{m} \cdot x$. Since both the vector field and phase portrait live in the $xv-$plane, it is a good idea to eliminate the time component. For this, compute $\frac{dv}{dx}$ using the chain rule to get $\frac{dv}{dt} \cdot \frac{dt}{dx} = \frac{-k}{m} \cdot \frac{x}{v}$. This is a separable differential equation!

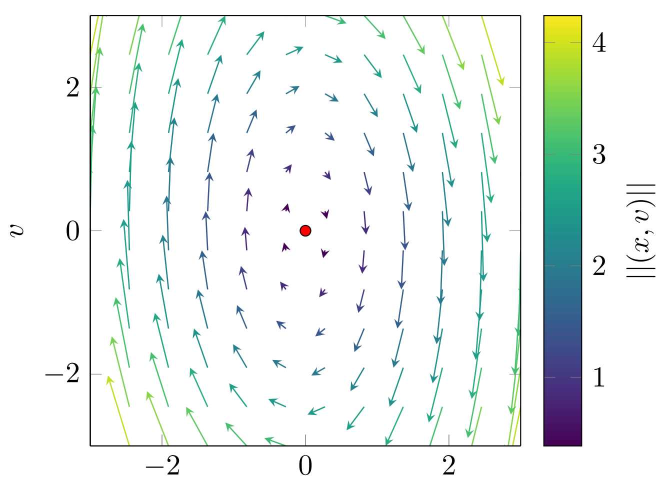
$$v\ dv = -\frac{k}{m} \cdot x\ dx \overset{\int \cdots}{\Longrightarrow} \frac{v^2}{2} = \frac{-k}{m} \cdot \frac{x^2}{2} + C \overset{\times 2}{\Longrightarrow} v^2 = -\frac{k}{m} \cdot x^2 + 2C \overset{+\frac{k}{m} \cdot x^2}{\Longrightarrow} v^2 + \frac{k}{m} \cdot x^2 = 2C$$
$$\overset{\times \frac{m}{2}}{\Longrightarrow} \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = mC$$

For physicists, this should be familiar! The expression $\frac{1}{2} \cdot mv^2$ encodes potential energy, whereas kinetic energy is displayed as $\frac{1}{2} \cdot kx^2$. This is the conservation law of energy. To see this, set $E = mC$ and write

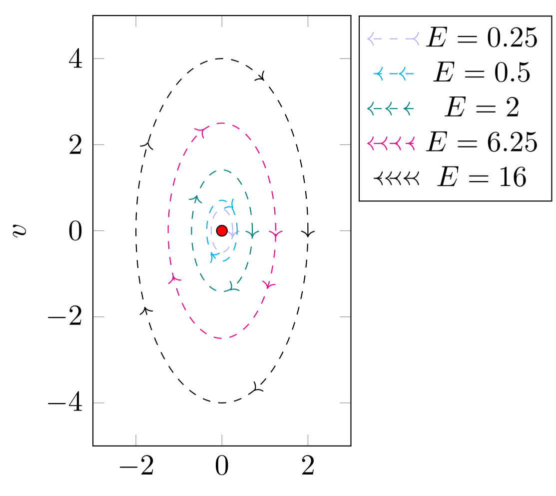
$$\begin{aligned} \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t) &= E(t) \\ &= E(0) := \frac{1}{2}mv^2(0) + \frac{1}{2}kx^2(0). \end{aligned}$$

The equation $E(t) = E(0)$ encodes that the initial total energy $E(0)$ is preserved as time flows. All points of the solutions $\binom{x}{v}$ with initial condition $\binom{x_0}{v_0}$ should therefore lie on an ellipse.[‡]

Vector Field Sketch



Phase Portrait



To study the change in phase portrait upon varying k, consider the horizontal endpoints of the ellipse on the $x-$axis. There, we have $v = 0$ and $E = \frac{1}{2}kx^2$. The energy is constant[§], therefore increasing k implies that x^2 must decrease. This forces both endpoints to get closer to the origin. You may convince yourself by a similar argument that increasing m shifts the ellipse in the vertical direction. $\boxed{\xi}$

[‡]The conservation law of energy is an equation of an ellipse. Do you see this?
[§]by the conservation law

Exercise 2 *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{\frac{2}{3}}$.*

Before commencing with the proof, let us give two precise definitions.[¶]

Definition 1.13 (Global Lipschitz Continuity)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *locally* Lipschitz continuous if there exists an $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}$.

Definition 1.13 (Local Lipschitz Continuity)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *globally* Lipschitz continuous if for every x_0 we may find a neighbourhood $\mathcal{U}_{x_0}$ around it such that

$$|f(x) - f(y)| \leq L_{x_0}|x - y|$$

for all $x, y \in \mathcal{U}_{x_0}$. The subscript $_{x_0}$ signifies the dependence of L on x_0.

Next, proceed to prove the following statements.

---

a) *Show that f is not locally Lipschitz continuous.*

Proof. First, notice that $f(x) := x^{\frac{2}{3}} = \sqrt[3]{x^2}$ behaves not-so-nicely near $x = 0$. Formally, the derivative

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

exists for $x \neq 0$, and is unbounded[‖] as x approaches 0. This makes it a possible candidate point to exploit. Proceed, and suppose for the sake of contradiction that f is Lipschitz. Then this suggests that we may find $\delta,\ L_{x_0} > 0$ such that

$$|x^{\frac{2}{3}} - y^{\frac{2}{3}}| \leq L_{x_0}|x - y|.$$

for all $x, y \in (-\delta, \delta)$. To utilise our earlier observation, set $y = 0$ and let $x \to 0^+$ to get

$$x^{\frac{2}{3}} \leq L_{x_0}x \overset{\times x^{-1}}{\Longrightarrow} x^{-\frac{1}{3}} \leq L_{x_0} \tag{*}$$

for some constant L. Notice, however, that (*) implies that

$$\infty = \lim_{x \to 0^+} x^{-\frac{1}{3}} \leq L_{x_0}. \quad \boxed{\lightning}$$

Clearly, there is no constant L_{x_0} that works, thus the assumption fails. $\boxed{\xi}$

[¶]Do you remember this from Analysis I? If not, seek Page 1, Lecture 8

[‖]$|f'(x)| \to \infty$

---

b) *Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$. Show that g is locally Lipschitz continuous but not globally Lipschitz continuous.*

Proof. To show that g is locally Lipschitz continuous, choose $x_0 \in \mathbb{R}$, and let $\mathcal{U}_{x_0} = [x_0 - \delta, x_0 + \delta]$ for some positive δ. Notice that Lipschitz continuity is equivalent to

$$|g(x) - g(y)| \leq L_{x_0}|x - y| \iff \frac{|g(x) - g(y)|}{|x - y|} \leq L_{x_0}.$$

Proceed with yet another observation $-$ g is continuous on the closed interval $[x_0 - \delta, x_0 + \delta]$ and differentiable on the open interval $(x_0 - \delta, x_0 + \delta)$. Apply the mean value theorem to establish the existence of some $\xi \in [x_0 - \delta, x_0 + \delta]$ for which

$$g'(\xi) = \frac{|g(x) - g(y)|}{|x - y|}$$

for all $x, y \in [x_0 - \delta, x_0 + \delta]$. Combining both facts, we notice that

$$\frac{|g(x) - g(y)|}{|x - y|} = g'(\xi) \leq L_{x_0}.$$

To bound the derivative $g'(x) = 3x^2$ over $[x_0 - \delta, x_0 + \delta]$, we note that $\max(|x_0 - \delta|, |x_0 + \delta|)$ maximizes g. Therefore, the choice of

$$L_{x_0} := \max(|x_0 - \delta|, |x_0 + \delta|)$$

gives the result. Observe how L always depends on the choice of x_0. In that respect, it is not universal.

In a style similar to **a)**, we show that g is not globally Lipschitz. Assume for the sake of contradiction that g is globally Lipschitz, then we establish the existence of L for which

$$|x^3 - y^3| \leq L|x - y|.$$

for any $x, y \in \mathbb{R}$. Therefore, it would not cause an issue if one makes the choice of $y = 0$ to get that

$$|x^3| \leq L \cdot |x| \overset{\times|x^{-1}|}{\Longrightarrow} |x^2| \leq L.$$

The implication is clearly false, since the statement should hold for all $x \in \mathbb{R}$. Taking $|x| \longrightarrow \infty$ gives the contradiction. ↯

Exercise 3 *Prove that every continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous. Hint: One possibility is to use the inequality $\left|\int_x^y g(t)\,dt\right| \leq \int_x^y |g(t)|\,dt$.*

The extreme value theorem in higher dimensions is stated later in Lectures 14 & 15.

Theorem 2.24 (Extreme Value Theorem)**.** *If $f : S \to \mathbb{R}^d$ is continuous on a closed and bounded set $S \subseteq \mathbb{R}^d$, then f attains a minimum and maximum value on S. Precisely, one writes*

$$(\forall \vec{x} \in S)(\exists\ \vec{a}, \vec{b} \in S)\ :\ f(\vec{a}) \leq f(\vec{x}) \leq f(\vec{b}).$$

It is a good idea nevertheless to include it here, as it recaps the one-dimensional case.

Proof. Given x_0, the goal is to show that we can find a neighbourhood $\mathcal{U}_{x_0}$ on which the Lipschitz condition is satisfied. Assume indeed that f is continuously differentiable. Then, f' is continuous, and for $x_0 \in \mathbb{R}$ we may choose $\mathcal{U}_{x_0} = [x_0 - \delta, x_0 + \delta]$ for some $\delta > 0$. Theorem 2.24 gives an upper bound

$$|f'(t)| \leq M_{x_0} \tag{*}$$

for every $t \in \mathcal{U}_{x_0}$. Notice that for fixed δ, the value of M_{x_0} depends on x_0. Next, write

$$f(y) - f(x) = \int_x^y f'(t)\ dt \qquad \text{(fundamental theorem of calculus)}$$

for $x, y \in \mathcal{U}_{x_0}$. We are yet to utilize the hint. To account for this shortcoming, write

$$|f(y) - f(x)| = \left|\int_x^y f'(t)\ dt\right| \leq \int_x^y |f'(t)|\ dt \overset{(*)}{\leq} M_{x_0} \cdot (y - x) \leq M_{x_0} \cdot |x - y|.$$

This statement is true for all $x, y \in \mathcal{U}_{x_0}$ with $x \leq y$. The point x_0 was arbitrarily chosen, therefore f must be locally Lipschitz. $\boxed{\xi}$

Sheet Four

Keywords

Euclidean Metrics, Triangle Inequality, Eigenvalues, Eigenvectors, Characteristic Equation, Gaussian Elimination, Augmented Matrices.

Contents

Exercise 1 *Let the functions $d_j : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ with $j \in \{1, 2, \infty\}$ be defined by*

- $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$,
- $d_2(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$,
- $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

---

a) *Show that all three functions define a metric on $\mathbb{R}^2$.*

Definition 1.15 (Section 1.5)**.** * Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ be a map into the positive reals. We say that d is a metric if it is

0. $d(x, y) = 0 \iff x = y$ *positive-definite*
1. $d(x, y) = d(y, x)$ *symmetric*
2. $d(x, y) + d(y, z) \geq d(x, z)$ *triangle-inequality*

for all $x, y, z \in \mathbb{R}$.

Proof. We proceed metric-by-metric.

- $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$. We would like to show that

  0. $d_1(x, y) = 0 \iff x = y$. This is easy, since

$$d_1(x, y) = 0 \iff \overbrace{|x_1 - y_1|}^{\geq 0} + \underbrace{|x_2 - y_2|}_{\geq 0} \iff |x_1 - y_1| = 0 = |x_2 - y_2| \iff x_1 = y_1 \,\&\, x_2 = y_2$$
$$\iff x = y.$$

  1. $d_1(x, y) = d_1(y, x)$. This follows from $|x| = -|x|$, since

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d_2(x, y)$$

  2. $d_1(x, y) + d_1(y, z) \geq d(x, z)$. The left-hand expression is given by

$$\begin{aligned} d_1(x, y) + d_1(y, z) &:= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &= \underbrace{|x_1 - y_1| + |y_1 - z_1|}_{\geq |x_1 - z_1|} + \overbrace{|x_2 - y_2| + |y_2 - z_2|}^{\geq |x_2 - z_2|} \qquad (1 - \Delta) \\ &\geq |x_1 - z_1| + |x_2 - z_2| =: d_1(x, z) \end{aligned}$$

---

*See Page 10, Lectures 6 & 7

where we make use of the $1-$ *triangle inequality* ($1-\Delta$). Let us justify its usage with a proof.

Lemma ($1-\Delta$)**.** *The inequality* $|a-b|+|b-c| \geq |a-c|$ *holds for all* $a,b,c \in \mathbb{R}$.

Proof. First, $|a|$ is defined to be $\max\{a,-a\}$. With this, write

$$\begin{cases} a \leq |a| \quad \text{and} \quad b \leq |b| & \implies a+b \leq |a|+|b| \\ -a \leq |a| \quad \text{and} \quad -b \leq |b| & \implies -a-b \leq |a|+|b| \end{cases} \implies |a+b| \leq |a|+|b|.$$

Applying this observation, we get that $|a-b|+|b-c| \geq |a\overbrace{-b+b}^{:=0}-c| = |a-c|$. Finally, note that the *addition of zero* trick is common, and in fact quite useful in various contexts. $\boxed{\xi}$

- $d_2(x,y) = \sqrt{|x_1-y_1|^2+|x_2-y_2|^2}$. Let us indeed demonstrate the following.

  0. $d_2(x,y)=0 \iff x=y$. Starting with $d_2(x,y)=0$, we get

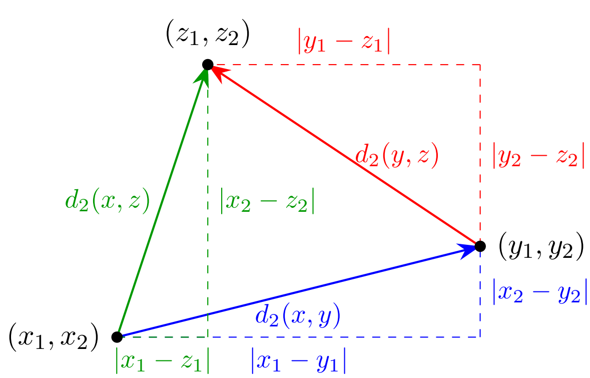
  $$\overbrace{\sqrt{|x_1-y_1|^2+|x_2-y_2|^2}}^{\geq 0} = 0 \iff \overbrace{|x_1-y_1|^2}^{\geq 0} + \overbrace{|x_2-y_2|^2}^{\geq 0} = 0 \iff |x_1-y_1| = 0 = |x_2-y_2|$$
  $$\iff x_1 = y_1 \ \& \ x_2 = y_2$$
  $$\iff x = y.$$

  1. $d_2(x,y) = d_2(y,x)$. Using $|-x| = |x|$ once again, we get

  $$d_2(x,y) := \sqrt{|x_1-y_1|^2+|x_2-y_2|^2} = \sqrt{|y_1-x_1|^2+|y_2-x_2|^2} := d_2(y,x).$$

  2. $d_2(x,y)+d_2(y,z) \geq d_2(x,z)$. You might have already noticed that d_2 is the Euclidean metric in $\mathbb{R}^2$. Geometrically, $d_2(x,y)$ measures the line $\overline{xy}$ by means of the Pythagorean theorem.

  Triangle Inequality



  We know from Euclidean geometry that $d_2(x,z)$ should not be greater than $d_2(x,y)+d_2(y,z)$. **This, however, is not sufficient (!) for a complete solution.** Instead, one must proceed as follows.

Lemma (2-Δ)**.** *The triangle inequality $d_2(x,y) + d_2(y,z) \geq d_2(x,z)$ holds for all $x, y, z \in \mathbb{R}^2$.*

Proof. To show that $d(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ satisfies the triangle inequality, it requires some effort. The Euclidean metric, is *translation-invariant*, meaning that

$$d_2(x,y) = d_2(x - z, y - z)$$

for $x, y, z \in \mathbb{R}^2$. This is not too difficult to check either. This reduces the statement to †

$$d_2(x,y) + d_2(y,z) \geq d_2(x,z) \iff d_2(x,0) + d_2(0,y) \geq d_2(x,y).$$

Geometrically, this is simply saying that we translate the triangle to the origin. The strategy for this proof is to reduce the statement to one that we can easily prove. Let us proceed in this sense, and write

$$\begin{aligned}
d_2(x,0) + d_2(0,y) \geq d_2(x,y) &\iff \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2} \geq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}. \\
\overset{(\,\cdot\,)^2}{\iff} \cancel{x_1^2 + x_2^2 + y_1^2 + y_2^2} + 2\sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)} &\geq (x_1 - y_1)^2 + (x_2 - y_2)^2 \\
&= \cancel{x_1^2 + x_2^2 + y_1^2 + y_2^2} - 2x_1y_1 - 2x_2y_2 \\
&\overset{\times\frac{1}{2}}{\iff} \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)} \geq -x_1y_1 - x_2y_2 \\
&\overset{(\,\cdot\,)^2}{\iff} \cancel{x_1^2y_1^2} + x_1^2y_2^2 + x_2^2y_1^2 + \cancel{x_2^2y_2^2} \geq \cancel{x_1^2y_1^2} + \cancel{x_2^2y_2^2} + 2x_1y_1x_2y_2 \\
&\iff x_1^2y_2^2 - 2x_1y_1x_2y_2 + x_2^2y_1^2 \geq 0
\end{aligned}$$

This is great, because the expression on the left-hand side is precisely $(x_1y_2 - x_2y_1)^2$, which is indeed always greater or equal to zero. With this we conclude the argument. $\boxed{\xi}$

- $d_\infty(x,y) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$. We show once more that

  0. $d_\infty(x,y) = 0 \iff x = y$. Start with the definition,

  $$d_\infty(x,y) = 0 \iff: \max\{\overbrace{|x_1 - y_1|}^{\geq 0}, \underbrace{|x_2 - y_2|}_{\geq 0}\} = 0 \iff |x_1 - y_1| = |x_2 - y_2| = 0 \iff x = y.$$

  1. $d_\infty(x,y) = d_\infty(y,x)$. Simply write

  $$d_\infty(x,y) := \max\{|x_1 - y_1|, |x_2 - y_2|\} = \max\{|y_1 - x_1|, |y_2 - x_2|\} =: d_\infty(y,x).$$

  2. $d_\infty(x,y) + d_\infty(y,z) \geq d_\infty(x,z)$. Start with $d_\infty(x,z) = \max\{|x_1 - z_1|, |x_2 - z_2|\}$ and notice that the 1−triangle inequality gives

  $$\begin{cases} |x_1 - z_1| & \leq |x_1 - y_1| + |y_1 - z_1| \\ |x_2 - z_2| & \leq |x_2 - y_2| + |y_2 - z_2| \end{cases} \implies d_\infty(x,z) \leq d_\infty(x,y) + d_\infty(y,z). \quad \boxed{\xi}$$

---

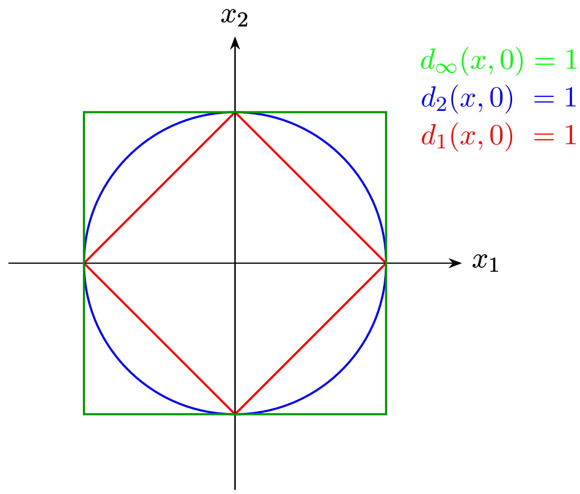
† To see this, notice that $d_2(x,y) = d_2(x - y, 0)$ and $d_2(y,z) = d_2(0, z - y)$ allow us to apply the *reduced* inequality to get $d_2(x - y, z - y)$. This in turn equals $d_2(x,z)$ by translating with y.

---

b) *For all three cases $j \in \{1, 2, \infty\}$, draw the set of points $\{x \in \mathbb{R}^2 : d_j(x, 0) = 1\}$.*

Drawing. The set of points is given by

Metrics $d_j(x, 0)$ for $|x| \le 1$



In the case of $d_1(x, 0) = 1$, we are considering $|x_1| + |x_2| = 1$. This is a linear relation. Taking care of cases where signs of x_1, x_2 switch, it gives this rhombus-shaped drawing. As for the event $d_2(x, 0)$, we have

$$\sqrt{|x_1 - 0|^2 + |x_2 - 0|^2} = \sqrt{x_1^2 + x_2^2} = 1,$$

that is the well-known equation of a circle. Finally, $d_\infty(x, 0) = 1$ gives $\max(|x_1|, |x_2|) = 1$, which in turn implies that $|x_1|$ or $|x_2|$ must be equal to one. In the case where both $|x_1| = |x_2| = 1$, we get the corners of this square. $\boxed{\xi}$

Exercise 2 *We consider the matrix* $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

a) *Compute the eigenvalues and the corresponding eigenvectors of the matrix A.*

Proof. Let λ be an eigenvalue of A. Then, λ is defined such that it satisfies $A\vec{v} = \lambda\vec{v}$. Proceed to obtain the characteristic equation

$$A\vec{v} - \lambda\vec{v} = 0 \implies (A - \lambda \cdot I_{2\times 2}{}^{\ddagger})\vec{v} = 0 \overset{\text{if}}{\underset{\vec{v}\neq 0}{\implies}} \det(A - \lambda \cdot I_{2\times 2}) = 0.$$

The matrix $A - \lambda \cdot I_{2\times 2} := \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$ gives a determinant of

$$\det(A - \lambda \cdot I_{2\times 2}) := (2-\lambda)^2 - 1 = 0 \implies 2 - \lambda = \pm 1 \implies \lambda_{1,2} = 2 \pm 1.$$

Next, the eigenvectors are obtained by solving the characteristic equation using the respective eigenvalue.

$$\overbrace{\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}}^{:=A-\lambda\cdot I_{2\times 2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \iff \begin{pmatrix} 2-\lambda \\ 1 \end{pmatrix} \cdot v_1 + \begin{pmatrix} 1 \\ 2-\lambda \end{pmatrix} \cdot v_2 = 0 \iff \begin{pmatrix} 2-\lambda \\ 1 \end{pmatrix} \cdot v_1 = \begin{pmatrix} -1 \\ \lambda - 2 \end{pmatrix} \cdot v_2$$

$$\iff \underbrace{\begin{cases} v_1 = v_2 \\ v_2 = s \in \mathbb{R}\backslash\{0\} \end{cases}}_{(\boldsymbol{\lambda_1 = 3})} \quad \& \quad \overbrace{\begin{cases} v_1 = -v_2 \\ v_2 = t \in \mathbb{R}\backslash\{0\} \end{cases}}^{(\boldsymbol{\lambda_2 = 1})}$$

$$\iff \vec{v}_1 = s \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \& \quad \vec{v}_2 = t \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For the sake of this discussion, we choose the pair $\vec{v}_1 = \binom{1}{1}$, $\vec{v}_2 = \binom{-1}{1}$. Note, however, that any pair of vectors $\vec{v}_1, \vec{v}_2$ of the prescribed form are perfectly suitable as eigenvectors of the matrix A. $\boxed{\xi}$

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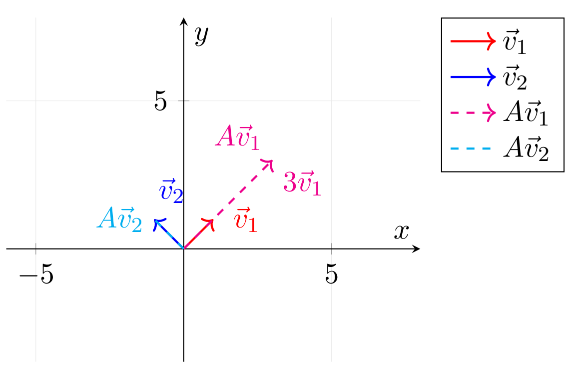
b) *Draw the eigenvectors on the two-dimensional plane. Interpret geometrically how the eigenvectors behave when the matrix A acts on them.*

Eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$



Sketch. The matrix has the effect of scaling each eigenvector by its eigenvalue. As shown earlier, applying A to $\vec{v}_1$ scales it by its associated eigenvalue, $\lambda_1 = 3$. The same may be said about $\vec{v}_2$, which is scaled by $\lambda_2 = 1$, i.e. fixed in place. $\boxed{\xi}$

‡ $I_{2\times 2}$ is the 2×2 identity matrix.

Exercise 3 *Compute all eigenvalues and eigenvectors of the matrix* $B = \begin{pmatrix} 3 & 1 & -2 \\ -2 & 0 & 4 \\ 1 & 1 & 0 \end{pmatrix}$.

Computation. As with (**iv.2**), we get a characteristic equation $\det(B - \lambda \cdot I_{3\times 3}) = 0$ for

$$\begin{aligned} \det(B - \lambda \cdot I) := \det \begin{pmatrix} 3-\lambda & 1 & -2 \\ -2 & -\lambda & 4 \\ 1 & 1 & -\lambda \end{pmatrix} &= (3-\lambda) \cdot \begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} -2 & 4 \\ 1 & -\lambda \end{vmatrix} + (-2) \cdot \begin{vmatrix} -2 & -\lambda \\ 1 & 1 \end{vmatrix} \\ &= (3-\lambda)(\lambda^2 - 4) - (2\lambda - 4) + (-2) \cdot (-2 + \lambda) \\ &= 3\lambda^2 - 12 - \lambda^3 \cancel{+4\lambda} \cancel{-2\lambda} + 4 + 4 \cancel{-2\lambda} \\ &= \underbrace{-\lambda^3 + 3\lambda^2 - 4}_{:=P(\lambda)}. \end{aligned}$$

Setting $P(\lambda) = 0$ forces us to find the roots of a degree-3 polynomial. We know how to proceed in the degree-2 case, so let us direct our focus on reducing P by one degree. The general strategy is as follows.

---

1. *Guess a root* λ_1. Usually, -2, -1, 0, 1, 2 are good guesses.
2. *Divide by* $(\lambda - \lambda_1)$. The Fundamental Theorem of Algebra tell us we can express P in terms of its roots,
$$P(\lambda) = (\lambda - \lambda_1) \cdot (\lambda - \lambda_2) \cdot (\lambda - \lambda_3).$$
The quotient is accordingly guaranteed to be a degree-2 polynomial.
3. *Apply the quadratic formula to obtain* λ_2, λ_3.

With this, one proceeds to make an educated guess of $\lambda_1 = -1$, which indeed gives $P(\lambda_1) = 0$. Next, divide P by $(\lambda - \lambda_1)$,[§]

$$\begin{array}{r|l} & -\lambda^2 + 4\lambda - 4. \\ \hline \lambda + 1 & -\lambda^3 + 3\lambda^2 \qquad - 4 \\ & \underline{\lambda^3 + \lambda^2} \\ & 4\lambda^2 \\ & \underline{-4\lambda^2 - 4\lambda} \\ & -4\lambda - 4 \\ & \underline{4\lambda + 4} \\ & 0 \end{array}$$

[§] Here, we use polynomial long division. Some people prefer synthetic division. Regardless of the method, our focus here is on ideas and not techniques.

The remaining eigenvalues are retrieved by setting $\overbrace{-\lambda^2 + 4\lambda - 4}^{-(\lambda-2)^2}$ to zero, giving $\lambda = 2$. We write

$$P(\lambda) = (\lambda+1)(\lambda-2)^2 = 0 \implies \lambda_1 = -1,\ \lambda_2 = 2.$$

With λ_2 *counted twice* as a root, we say that $\lambda_2 = 2$ has multiplicity 2. Next the associated set of eigenvectors is obtained by solving

$$\overbrace{\begin{pmatrix} 3-\lambda & 1 & -2 \\ -2 & -\lambda & 4 \\ 1 & 1 & -\lambda \end{pmatrix}}^{A-\lambda\cdot I} \overbrace{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}}^{\vec{v}} = 0 \quad \text{for } \vec{v} \neq \tilde{\mathbf{0}}.$$

This will require some effort. Let us express this system in terms of an augmented matrix, and perform row operations to obtain a solution.

$$\begin{pmatrix} 3-\lambda & 1 & -2 \\ -2 & -\lambda & 4 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \overset{\text{augmented}}{\underset{\text{matrix}}{\iff}} \left(\begin{array}{ccc|c} 3-\lambda & 1 & -2 & 0 \\ -2 & -\lambda & 4 & 0 \\ 1 & 1 & -\lambda & 0 \end{array}\right)$$

The solution row is zero, therefore applying row operations would keep it zero. Proceed with Gaussian elimination for each eigenvalue.

$$\left(\begin{array}{ccc|c} 4 & 1 & -2 & 0 \\ -2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \mid \tfrac{1}{4} = \left(\begin{array}{ccc|c} \boxed{1} & \frac{1}{4} & \frac{-1}{2} & 0 \\ -2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \begin{matrix} \cdot 2, \cdot(-1) \\ + \\ + \end{matrix} = \left(\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{-1}{2} & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ 0 & \frac{3}{4} & \frac{3}{2} & 0 \end{array}\right) \mid \tfrac{2}{3} \qquad (\boldsymbol{\lambda_1 = -1})$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{-1}{2} & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & \frac{3}{4} & \frac{3}{2} & 0 \end{array}\right) \begin{matrix} + \\ \cdot\frac{-1}{4}, \cdot\frac{-3}{4} \\ + \end{matrix} = \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\overset{\P}{=} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \iff \begin{cases} v_1 = v_3 \\ v_2 = -2v_3 \\ v_3 = s \in \mathbb{R} \end{cases} \iff \boxed{\vec{v}_1 = s \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \neq \vec{\mathbf{0}}.}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ -2 & -2 & 4 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \begin{matrix} \cdot 2, \cdot(-1) \\ + \\ + \end{matrix} = \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{\P}{=} \left(\begin{array}{c|ccc} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad (\boldsymbol{\lambda_2 = 2})$$

$$\iff \begin{cases} v_1 = -v_2 + 2v_3 \\ v_2 = s \in \mathbb{R} \\ v_3 = t \in \mathbb{R} \end{cases} \iff \boxed{\vec{v}_2 = s \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \neq \vec{\mathbf{0}}.}$$

¶ The terms are subtracted to the other side of the equation.

Next, some observations.

1. The eigenvector v_1 lives in a one-dimensional space, whereas v_2 lives in a two-dimensional space. This is related to the *multiplicity* of each eigenvalue, especially that λ_2 had multiplicity 2.

2. The action of A on every vector v in the respective *eigenspace* is exactly the same: it scales v by the respective eigenvalue. $\boxed{\xi}$

Sheet Five

Keywords

Stability of Solutions, Eigenvalue Analysis, Pointwise Continuity, Partial Derivatives, Gradient, Directional Derivatives, Direction of Steepest Change.

Contents

Exercise 1 *Consider the linear system*

$$\begin{cases} \frac{d}{dt}x(t) = -2x(t) + 4y(t) \\ \frac{d}{dt}y(t) = -x(t) - 3y(t). \end{cases}$$

Determine the stability of the stationary solution $(0,0)$*.*

Theorem 1.26 (Section 1.7, Lecture 9)**.** *Let A be an arbitrary matrix. Then, the stationary point $\tilde{\mathbf{0}}$ is an asymptotically stable solution of $\frac{d}{dt}x(t) = Ax(t)$ if and only if Re$(\lambda) < 0$ for all eigenvalues λ of A.**

*Remark.*For more on this, see Theorem 1.26, Section 1.7, Lecture 9.

Proof. Let us start by representing our system of differential equations in the language of matrices.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} -2 & 4 \\ -1 & -3 \end{pmatrix}}^{:=A} \begin{pmatrix} x \\ y \end{pmatrix}$$

To study the stability of our solutions, it is a good idea to study the eigenvalues of A.

$$\det(A - \lambda \cdot I) := \det \begin{pmatrix} -2-\lambda & 4 \\ -1 & -3-\lambda \end{pmatrix} = (-2-\lambda)(-3-\lambda) + 4 = \lambda^2 + 5\lambda + 10.$$

Setting the characteristic equation to zero, we get that

$$\lambda_{1,2} = \frac{1}{2}(-5 \pm \sqrt{25 - 4 \cdot 10 \cdot 1}) = \frac{1}{2}(-5 \pm i\sqrt{15}).$$

It is now possible to perform an eigenvalue analysis. Note that Re(λ_1), Re(λ_2) are both negative, indicating that $\tilde{\mathbf{0}}$ is asymptotically stable by Theorem 1.26. $\boxed{\xi}$

Exercise 2 *Prove that the function* $f : \mathbb{R}^3 \to \mathbb{R}$ *defined by* $(x, y, z) \mapsto x + 2y^2 + 3z^3$ *is continuous at every point in* $\mathbb{R}^3$.

Definition 2.4 (Section 2.1, Lectures 10 & 11)**.** A function $f : \mathcal{U} \to \mathbb{R}$ ($\mathcal{U} \subseteq \mathbb{R}^d$ open) is continuous at $\vec{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$ if

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a}).$$

One may reformulate the previous statement as

$$\lim_{n\to\infty} |f(\vec{x}_n) - f(\vec{a})| = 0$$

for any sequence $\{\vec{x}_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ that converges to $\vec{a}$.[†]

Proof. In the same spirit of the lecture example[‡], choose an arbitrary sequence (x_n, y_n, z_n) in $\mathbb{R}^3$ that converges to a point $\vec{a} := (x, y, z)$ as n tends to infinity. Formally, we write this as

$$|(x_n, y_n, z_n) - (x, y, z)| \underset{n\to\infty}{\longrightarrow} 0.$$

This implies that the sequence converges coordinate-wise, i.e.

$$|x_n - x| \underset{n\to\infty}{\longrightarrow} 0 \qquad |y_n - y| \underset{n\to\infty}{\longrightarrow} 0 \qquad |z_n - z| \underset{n\to\infty}{\longrightarrow} 0. \tag{*}$$

Next, to show that f is continuous, we need to show that $|f(x_n, y_n, z_n) - f(x, y, z)| \underset{n\to\infty}{\longrightarrow} 0$. We do this step-by-step. The next obvious move is to use the definition of f and write

$$|(x_n + 2y_n^2 + 3z_n^3) - (x + 2y^2 + 3z^2)| = |(x_n - x) + 2(y_n^2 - y^2) + 3(z_n^3 - z^3)|.$$

The flavour of this proof (and similar ones) is to manipulate the above expression into something of the form (*). We already have the expression for $|x_n - x|$, however some diligence is due when dealing with the other terms. To overcome this difficulty, we utilise the formulas for differences of two squares and two cubes.

$$|(x_n - x) + 2(y_n^2 - y^2) + 3(z_n^3 - z^3)| = |(x_n - x) + 2(y_n - y)\underbrace{(y_n + y)}_{(!)} + 3(z_n - z)\underbrace{(z_n^2 + z_n z + z^2)}_{(!!)}|$$

[†]See Page 3, Section 2.1, Lectures 10 & 11

[‡]Example 1, Section 2.1, Page 4, Lectures 10 and 11.

Much better, but we still need to take care of (!), (!!) terms. Here, the trick is to add zero.

$$y_n + y = (y_n - y) + 2y \tag{!}$$

$$\begin{aligned} z_n^2 + z_n z + z^2 &= (z_n - z)^2 + 3z_n z \\ &= (z_n - z)^2 + 3(z_n - z + z)z \\ &= (z_n - z)^2 + 3(z_n - z) \cdot z + 3z^2. \end{aligned} \tag{!!}$$

We manipulated both expressions by adding and subtracting y or z, and are now ready to take the limit. This is because

$$\begin{aligned} &\left|(x_n + 2y_n^2 + 3z_n^3) - (x + 2y^2 + 3z^2)\right| = \left|(x_n - x) + 2(y_n^2 - y) + 3(z_n^3 - z)\right| = \\ =&\left|(x_n - x) + 2(y_n - y)\underbrace{(y_n + y)}_{(!)} + 3(z_n - z)\underbrace{(z_n^2 + z_n z + z^2)}_{(!!)}\right| \\ =&\left|(x_n - x) + 2(y_n - y)(y_n - y + 2y) + 3(z_n - z)\left[(z_n - z)^2 + 3(z_n - z) \cdot z + 3z^2\right]\right| \\ \le&\underbrace{|x_n - x|}_{\underset{n\to\infty}{\longrightarrow} 0} + 2 \cdot \underbrace{|y_n - y|^2}_{\underset{n\to\infty}{\longrightarrow} 0} + 4y\underbrace{|y_n - y|}_{\underset{n\to\infty}{\longrightarrow} 0} + 3\underbrace{|z_n - z|^3}_{\underset{n\to\infty}{\longrightarrow} 0} + 9z \cdot \underbrace{|z_n - z|^2}_{\underset{n\to\infty}{\longrightarrow} 0} + 9z^2 \cdot \underbrace{|z_n - z|}_{\underset{n\to\infty}{\longrightarrow} 0} \end{aligned} \tag{1 − Δ}$$

all go to zero! Therefore, if we bound

$$|(x_n + 2y_n^2 + 3z_n^3) - (x + 2y^2 + 3z^2)| \underset{n\to\infty}{\le} 0$$

then the function is indeed continuous. To conclude, we utilised the assumption (*) as well as some clever manipulations to show that

$$\lim_{n\to\infty} |f(\vec{x}_n) - f(\vec{a})| = 0$$

for arbitrarily chosen $\vec{a}$, $\{\vec{x}_n\}$. $\boxed{\xi}$

Exercise 3 *Compute the partial derivatives of the following functions.*

---

f. $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = x^2 y + e^x \sin(y)$

Computation. We would like to compute the two possible partial derivatives, namely $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Let us start with $\frac{\partial f}{\partial x}$. To proceed, we fix[§] y and differentiate with respect to x only.

$$\begin{aligned}\frac{\partial f}{\partial x} := \frac{\partial}{\partial x}\left(x^2 y + e^x \sin y\right) \overset{\text{linearity}}{=} \frac{\partial}{\partial x} x^2 y + \frac{\partial}{\partial x} e^x \sin y &\overset{\text{fixing } y}{=} y \cdot \frac{d}{dx} x^2 + \sin y \cdot \frac{d}{dx} e^x \\ &= y \cdot 2x + \sin y \cdot e^x.\end{aligned}$$

Notice that since y is fixed, it is insignificant when computing the partial derivative with respect of x. The partial derivative becomes a total derivative on x only. With this clarified, we compute the remaining partial derivative to get

$$\begin{aligned}\frac{\partial f}{\partial y} := \frac{\partial}{\partial y}\left(x^2 y + e^x \sin y\right) \overset{\text{linearity}}{=} \frac{\partial}{\partial y} x^2 y + \frac{\partial}{\partial y} e^x \sin y &\overset{\text{fixing } x}{=} x^2 \cdot \frac{d}{dy} y + e^x \cdot \frac{d}{dy} \sin y \\ &= x^2 + e^x \cos y.\end{aligned}$$

$\boxed{\xi}$

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g. $g : \mathbb{R}^3 \to \mathbb{R}$ *defined by* $g(x, y, z) = \ln(x + y^2 + z^3)$.

Computation. Here we have three possible partial derivatives. As seen in the previous exercise, one differentiates only with respect to the partial derivative, fixing all other remaining variables. Let us immediately proceed with the computation.

To simplify[¶], let t be one of the three variables x, y, z. Then, notice that

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left[\ln(x + y^2 + z^3)\right] = \frac{1}{x + y^2 + z^3} \cdot \frac{\partial}{\partial t}(x + y^2 + z^3) \qquad \text{(One-variable Chain Rule)}$$

the computation simplifies to that of computing the partial derivatives of the inside function $x + y^2 + z^3$.

$$\frac{\partial}{\partial x}(x + y^2 + z^3) = 1 \qquad \frac{\partial}{\partial y}(x + y^2 + z^3) = 2y \qquad \frac{\partial}{\partial z}(x + y^2 + z^3) = 3z^2.$$

With this, we may write

$$\frac{\partial f}{\partial x} = \frac{1}{x + y^2 + z^3} \cdot 1 \qquad \frac{\partial f}{\partial y} = \frac{1}{x + y^2 + z^3} \cdot 2y \qquad \frac{\partial f}{\partial z} = \frac{1}{x + y^2 + z^3} \cdot 3z^2$$

$\boxed{\xi}$

[§]i.e. treat as constant

[¶]simplify=not write the same expression three times...

---

h. *$h : \mathbb{R}^2 \to \mathbb{R}$ defined by* $h(x,y) = \dfrac{xy}{x^2+y^2+1}$.

Computation. First, notice that h is symmetric with respect to its two inputs.[‖] This is great, since it allows us to compute the partial with respect to x, then interchange its variables to get the partial with respect to y. For the sake of conciseness, let $\partial_x h$ denote the partial derivative of h with respect to x.

$$\begin{aligned}\partial_x h = \partial_x\left(\frac{xy}{x^2+y^2+1}\right) = y\cdot\partial_x\left(\frac{x}{x^2+y^2+1}\right) &= y\cdot\frac{(x^2+y^2+1)\cdot(\partial_x\ x) - [\partial_x\ (x^2+y^2+1)]\cdot x}{(x^2+y^2+1)^2}\\ &= y\cdot\frac{\cancel{x^2}+y^2+1-\cancel{2}x^2}{(x^2+y^2+1)^2}\\ &= y\cdot\frac{-x^2+y^2+1}{(x^2+y^2+1)^2}.\end{aligned}$$

By symmetry of h, we get that $(\partial_y h)(x,y) = (\partial_x h)(y,x) = x\cdot\dfrac{-y^2+x^2+1}{(y^2+x^2+1)^2}$. $\boxed{\xi}$

Exercise 3.1. Check that $(\partial_y h)(x,y) = x\cdot\dfrac{-y^2+x^2+1}{(y^2+x^2+1)^2}$!

[‖] i.e. $h(x,y) = h(y,x)$.

Exercise 4 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $(x, y) = e^x \sin(y) + y^3 \ln(x)$.*

Theorem 2.12 (Section 2.4, Lectures 10 & 11)**.** *If f is differentiable at $\vec{a}$, then the directional derivatives of f at $\vec{a}$ all exist, and are given by $\partial_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$.* **

Corollary *$(\vec{\nabla} f)(\vec{a})$ points in the direction of steepest increase of f at $\vec{a}$. Moreover, the direction of no increase is that perpendicular to the direction of $(\vec{\nabla} f)(\vec{a})$.*

Proof. A quick justification for this is that the dot-product formula given by

$$\nabla f(\vec{a}) \cdot \vec{u} = |\nabla f(\vec{a})||\vec{u}| \cos \theta_{(\nabla f(\vec{a}), \vec{u})}.$$

The increase is clearly maximal for $\theta_{(\nabla f(\vec{a}), \vec{u})} = 0$, when $\vec{u}$ points in the same direction as$(\vec{\nabla} f)(\vec{a})$. On the other extreme, there is no increase in the case $\theta_{(\nabla f(\vec{a}), \vec{u})} = \frac{\pi}{2}$; that is, when $\vec{u} \perp \nabla f(\vec{a})$. $\boxed{\xi}$

---

a) *Determine the direction in which f experiences the largest change at the point $(1, 1)$.*

Solution. The question asks to compute the *gradient* of f at $(1, 1)$. Simply, the gradient is given by

$$\nabla f(x, y) = (\partial_x f, \partial_y f) = (e^x \sin y + \frac{y^3}{x}, e^x \cos y + 3y^2 \ln x).$$

Evaluating at $(1, 1)$, it gives

$$\nabla f(1, 1) = (e \sin(1) + 1, e \cos 1).$$ $\boxed{\xi}$

---

b) *Determine the direction in which f does not change at the point $(1, 1)$.*

Solution. The direction of no change is that which is perpendicular to ∇f. The dot product formula portrays this direction as that of the vector $\vec{u}$ for which

$$\begin{aligned} \nabla f(1, 1) \cdot \vec{u} = 0 &\iff (e \sin(1) + 1, e \cos 1) \cdot \vec{u} = 0 \\ &\iff (e \sin(1) + 1) \cdot u_1 + (e \cos 1) \cdot u_2 = 0. \end{aligned} \qquad (*)$$

Then, it is then easy to see that the choices $u_1 = -e \cos 1$ and $u_2 = e \sin(1) + 1$ satisfy (*). Therefore, the direction

$$\vec{u} = (-e \cos(1), e \sin(1) + 1)$$

is that of no change at $(1, 1)$. $\boxed{\xi}$

** See Pages 13, 14, Section 2.4, Lectures 10 & 11.

Sheet Six

Keywords

Multi-variable Chain Rule, Taylor Polynomials, Multi-indices, Partial Derivatives, Mean Value Theorem, Schwartz Theorem, Limit Definition of a Derivative, Higher-Order Partial Derivatives.

Contents

Exericse 1. *Let* $f : \mathbb{R}^3 \to \mathbb{R}$ *be given by* $f(x, y, t) = x^2 + xy + y^2 + te^x$ *and* $(x(t), y(t)) = (e^t \cos(t), e^t \sin(t))$. *Compute* $\frac{d}{dt} f(x(t), y(t), t)$.

Solution. Let us present two methods to tackle this problem. For the sake of conciseness, we set $x(t) = x$ and $y(t) = y$ in both methods.

Direct Computation

Solution. We are asked to compute the derivative with respect to t of

$$f(t)^{*} := x^2 + xy + y^2 + te^x. \tag{*}$$

One may proceed by directly substituting the values for each function, and computing the expression

$$\begin{aligned} f(x, y, t) &:= [e^t \cos t]^2 + (e^t \cos t)(e^t \sin t) + [e^t \sin t]^2 + te^{e^t \cos t} \\ &= e^{2t} \underbrace{[\cos^2 t + \sin^2 t]}_{:=1} + e^{2t} \underbrace{\cos t \sin t}_{:=\frac{1}{2}\sin 2t} + te^{e^t \cos t} \\ &= e^{2t}[1 + \frac{1}{2} \sin 2t] + te^{e^t \cos t} \end{aligned}$$

which gives

$$\begin{aligned} \frac{d}{dt} f(x, y, t) &:= \frac{d}{dt} \left(e^{2t}[1 + \frac{1}{2} \sin 2t] + te^{e^t \cos t} \right) \\ &= e^{2t} \cos 2t + e^{2t}[2 + \sin 2t] + e^{e^t \cos t}(1 + te^t \cdot (\cos t - \sin t)) \\ &= e^{2t}(2 + \cos 2t + \sin 2t) + e^{e^t \cos t}(1 + te^t \cdot (\cos t - \sin t)) \end{aligned}$$

$\boxed{\xi}$

Using Chain Rule

Solution. Let us proceed differently, and apply the chain rule for the expression in (*).[†]

$$\begin{aligned} \frac{d}{dt} f(t) &:= \frac{d}{dt} \left(x^2 + xy + y^2 + te^x \right) \\ &= (2x \cdot x') + (x'y + xy') + (2y \cdot y') + (e^x + te^x \cdot x') \\ &= 2(x \cdot x' + y \cdot y') + (x'y + xy') + e^x(1 + t \cdot x'). \end{aligned}$$

If we can find expressions for x' and y', we are done.

$$\begin{aligned} x = e^t \cos t &\implies x' = e^t(\cos t - \sin t) \\ y = e^t \sin t &\implies y' = e^t(\sin t + \cos t) \end{aligned}$$

[*] The function f can be expressed in terms of t, since x, y are in essence functions of t.

[†] The product rule is used in the second term, with $(xy)' = x' \cdot y + y \cdot x'$.

Then, compute term-by-term.

$$\begin{cases} x \cdot x' = e^t \cos t \cdot e^t(\cos t - \sin t) = e^{2t}(\cos^2 t - \cos t \sin t) \\ y \cdot y' = e^t \sin t \cdot e^t(\sin t + \cos t) = e^{2t}(\sin^2 t + \cos t \sin t) \\ x' \cdot y = e^t(\cos t - \sin t) \cdot e^t \sin t = e^{2t}(\cos t \sin t - \sin^2 t) \\ x \cdot y' = e^t \cos t \cdot e^t(\sin t + \cos t) = e^{2t}(\cos t \sin t + \cos^2 t) \\ t \cdot x' = te^t(\cos t - \sin t) \end{cases}$$

Using these computations, we write

$$(1) \quad x' \cdot x + y' \cdot y = e^{2t}(\underbrace{\cos^2 t + \sin^2 t \cancelto{0}{- \cos t \sin t + \cos t \sin t}}_{=1}) = e^{2t}$$

$$(2) \quad x' \cdot y + x \cdot y' = e^{2t}(\cos t \sin t + \underbrace{\cos^2 t - \sin^2 t}_{:=\cos 2t} + \cos t \sin t) = e^{2t}(\underbrace{2 \cos t \sin t}_{:=\sin 2t} + \cos 2t) = e^{2t}(\sin 2t + \cos 2t)$$

And the two equations (1), (2) in turn give

$$\begin{aligned} \frac{d}{dt} f(x, y, t) &:= 2\overbrace{(x \cdot x' + y \cdot y')}^{(1)} + \overbrace{(x'y + xy')}^{(2)} + e^x(1 + t \cdot x') \\ &= 2e^{2t} + e^{2t}(\cos 2t + \sin 2t) + e^{e^t \cos t}(1 + te^t(\cos t - \sin t)) \\ &= e^{2t}(2 + \cos 2t + \sin 2t) + e^{e^t \cos t}(1 + te^t \cdot (\cos t - \sin t)). \end{aligned}$$

This conveniently matches the result previously attained. $\boxed{\xi}$

Exericse 2. *Find the second-order Taylor polynomial centered at* $(0,0)$ *of the function* $f:\mathbb{R}^2\to\mathbb{R}$ *given by* $f(x,y)=e^{xy}\sin(x+y)$.

Definition (Multi-Index Notation.)[‡] Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_d)$ be $d-$tuple of non-negative integers. Then,

1. $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_d$ — *Order or degree of* α
2. $\alpha!=\alpha_1!\alpha_2!\dots\alpha_d!$ — *Factorial is defined coordinate−wise*
3. $\partial^\alpha f=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_d^{\alpha_d}f:=\dfrac{\partial^{|\alpha|}f}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2\dots\partial^{\alpha_d}x_d}$ — *Describes different partial derivatives*
4. $\begin{pmatrix}v_1\\v_2\\\vdots\\v_d\end{pmatrix}^\alpha:=\begin{pmatrix}v_1^{\alpha_1}\\v_2^{\alpha_2}\\\vdots\\v_d^{\alpha_d}\end{pmatrix}$ — *Coordinate−wise powers*

For the sake of this discussion, let $f:\mathbb{R}^d\to\mathbb{R}$ be $k+1-$times continuous on an open convex set S, and choose a point $\vec{a}\in S$.

Theorem 2.18 (Taylor's Theorem in Several Variables)**.** *The* k^{th} *Taylor expansion of* $f(\vec{h})$ *around* $\vec{a}$ *is given by*

$$f(\vec{h}+\vec{a})\approx\sum_{|\alpha|\le k}\frac{\partial^\alpha f(\vec{a})}{\alpha!}\cdot\vec{h}^\alpha.$$

Lemma. *The second-order Taylor expansion of* f *around* $\vec{a}$ *is given by*

$$f(\vec{a}+\vec{h})\approx f(\vec{a})+\sum_{j=1}^d\partial_j f(\vec{a})\cdot h_j+\frac{1}{2}\sum_{j,k=1}^d\partial_{j,k}f(\vec{a})\cdot h_jh_k.$$

Proof. As part of Section 2.7, an argument is provided in page 10 of Lectures 12, 13. Here, $\partial_{j,k}$ is understood as taking the partial derivative with respect to the j^{th} and k^{th} variables. $\boxed{\xi}$

Solution. Since f has two variables x,y, we set $d=2$. The vector $\vec{h}$ of variables is $\vec{h}=(x,y)$, and thus the 2^{nd}-order Taylor polynomial of $f(x,y)$ around $\vec{a}=(0,0)$ should be given by

$$\begin{aligned}
f(\vec{h})&\overset{\text{(Lemma)}}{\approx} f(\vec{a})+\sum_{j=1}^d\partial_j f(\vec{a})\cdot h_j+\frac{1}{2}\sum_{j,k=1}^2\partial_{j,k}f(\vec{a})\cdot h_jh_k.\\
&=f(0,0)+\partial_{x_1}f(0,0)\cdot h_1+\partial_{x_2}f(0,0)\cdot h_2+\\
&\qquad+\frac{1}{2}\left(\partial_{x_1x_1}f(0,0)\cdot h_1h_1+\underbrace{\partial_{x_1x_2}f(0,0)\cdot h_1h_2+\partial_{x_2x_1}f(0,0)\cdot h_2h_1}_{\text{both terms are equal}}+\partial_{x_2x_2}f(0,0)\cdot h_2h_2\right)\\
&=f(0,0)+\partial_x f(0,0)\cdot x+\partial_y f(0,0)\cdot y+\\
&\qquad+\frac{1}{2}\left(\partial_{xx}f(0,0)\cdot x^2+2\cdot\partial_{xy}f(0,0)\cdot xy+\partial_{yy}f(0,0)\cdot y^2\right).
\end{aligned}\tag{*}$$

[‡]See section 2.7 of Lectures 12, 13.

We compute the following list of derivatives of $f(x, y) = e^{xy}\sin(x + y)$, which will be the coefficients of our Taylor polynomial.

$$\begin{aligned}
f &= e^{xy}\sin(x+y).\\
\partial_x f &= \partial_x e^{xy}\sin(x+y) = ye^{xy}\sin(x+y) + e^{xy}\cos(x+y) = e^{xy}\left[y\sin(x+y) + \cos(x+y)\right].\\
\partial_y f &= \partial_y e^{xy}\sin(x+y) = xe^{xy}\sin(x+y) + e^{xy}\cos(x+y) = e^{xy}\left[x\sin(x+y) + \cos(x+y)\right].\\
\partial_{xx} f &:= \partial_x \overbrace{\left[e^{xy}\left[y\sin(x+y) + \cos(x+y)\right]\right]}^{=\partial_x f}\\
&= ye^{xy}(y\sin(x+y) + \cos(x+y)) + e^{xy}(y\cos(x+y) - \sin(x+y)).\\
\partial_{yx} f &:= \partial_y \overbrace{\left(e^{xy}\left[y\sin(x+y) + \cos(x+y)\right]\right)}^{=\partial_x f}\\
&= xe^{xy}\left[y\sin(x+y) + \cos(x+y)\right] + e^{xy}\left[\sin(x+y) + y\cos(x+y) - \sin(x+y)\right].\\
\partial_{yy} f &:= \partial_y \overbrace{\left(e^{xy}\left[x\sin(x+y) + \cos(x+y)\right]\right)}^{=\partial_y f}\\
&= xe^{xy}\left[x\sin(x+y) + \cos(x+y)\right] + e^{xy}(x\cos(x+y) - \sin(x+y)).
\end{aligned}$$

Quick Exercise. Notice that in the computation of $\partial_{xy} f(x, y)$, we considered $\partial_{yx}\ f(x, y)$. Can you check that $\partial_{xy}\ f(x, y)$ yields the same result?

Next, compute these derivatives at the expansion point $(x, y) = 0$.

$$\begin{aligned}
f(0,0) &= e^{0\cdot 0}\cancelto{0}{\sin(0+0)} = 0.\\
\partial_x f(0,0) &= e^{0\cdot 0}\left[\cancelto{0}{0\sin(0+0)} + \cos(0+0)\right] = 1.\\
\partial_y f(0,0) &= e^{0\cdot 0}\left[\cancelto{0}{0\sin(0+0)} + \cos(0+0)\right] = 1\\
\partial_{xx} f(0,0) &= \cancelto{0}{0\cdot^{0\cdot 0}(0\sin(0+0) + \cos(0+0))} + e^{0\cdot 0}(\cancelto{0}{0\cos(0+0)} - \cancelto{0}{\sin(0+0))} = 0.\\
\partial_{yx} f(0,0) &= \cancelto{0}{0\cdot^{0\cdot 0}\left[0\sin(0+0) + \cos(0+0)\right]} + e^{0\cdot 0}\left[\cancelto{0}{\sin(0+0)} + \cancelto{0}{0\cos(0+0)} - \cancelto{0}{\sin(0+0)}\right] = 0.\\
\partial_{yy} f(0,0) &= \cancelto{0}{0\cdot^{0\cdot 0}\left[0\sin(0+0) + \cos(0+0)\right]} + e^{0\cdot 0}(\cancelto{0}{0\cos(0+0)} - \cancelto{0}{\sin(0+0)}) = 0.
\end{aligned}$$

It looks like only order-one terms *survive*. Equation (*) gives

$$f(x, y) \approx \partial_x f(0,0)\cdot x + \partial_y f(0,0)\cdot y = x + y.$$

With this, we announce $T^2_f(0,0) = \ x + y$ to be the second-order Taylor polynomial of f, centered at $(0, 0)$. From this tedious computation follows a simple result. $\boxed{\xi}$

Exercise 3 *Let $d \in \mathbb{N}$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a function that is continuously differentiable on an open set containing the line segment L between two points $\vec{a}, \vec{b} \in \mathbb{R}^d$. Prove the Mean Value Theorem for Several Variables, which states that: There exists a point $\vec{c}$ on L such that $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$.*

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1. *Define a function of a single variable $g : [0,1] \to \mathbb{R}$ by $g(t) = f(\vec{a} + t(\vec{b} - \vec{a}))$.*

Solution. First, consider the line segment L connecting $\vec{a}$ and $\vec{b}$. For simplicitly, think of the case of $d = 2$.

$$\vec{a} \underset{L}{\longleftrightarrow} \vec{b}$$

It is interesting to note that L is parameterized (described) by $\vec{a} + t(\vec{b} - \vec{a})$. For every point $l \in L$, there is a unique t for which $l = \vec{a} + t(\vec{b} - \vec{a})$. To see this, notice that the vector $\vec{b} - \vec{a}$ gives the direction, and $\vec{a}$ gives the starting point.

We may now think of g as mapping to L first, then L mapping to $\mathbb{R}$ via f. This is perfectly suitable, since $f : L \to \mathbb{R}$ is given to be continuously differentiable on L. Therefore, the function

$$g(t) := f(\vec{a} + t(\vec{b} - \vec{a}))$$

is continuously differentiable on $[0,1]$. $\boxed{\xi}$

---

2. *Apply the mean value theorem in One Variable to the function g.*

Solution. Perhaps it is a good idea to first state the theorem,

Theorem (Mean-value)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$, differentiable on the open interval (a,b). Then, there exists a $c \in (a,b)$ for which*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

To apply the mean value theorem on $g : [0,1] \to \mathbb{R}$, one must verify that g is continuous on $[0,1]$ and differentiable on $(0,1)$. This has already been checked in **(1.)**, therefore we may safely find $c \in (0,1)$ for which

$$g'(c) = \frac{g(1) - g(0)}{1 - 0} = g(1) - g(0) = f(\vec{b}) - f(\vec{a}). \tag{a}$$

We explore the consequences of this in the next sub-question. $\boxed{\xi}$

---

3. *Compute $\frac{d}{dt}g(t)$ using the chain rule and express it in terms of ∇f. Use this to conclude the proof by identifying a point $\vec{c}$ that satisfies the theorem.*

Proof. Applying the chain rule, the derivative of $g(t) := f(\vec{a} + t(\vec{b} - \vec{a}))$ is given by

$$g'(t) = \Big[\nabla f(\vec{a} + t(\vec{b} - \vec{a}))\Big] \cdot \underbrace{\Big[(\vec{a} + t(\vec{b} - \vec{a}))\Big]'}_{=\vec{b}-\vec{a}} = \nabla f(\vec{a} + t(\vec{b} - \vec{a})) \cdot (\vec{b} - \vec{a}). \tag{b}$$

Combining this with the result from (**2.**), we may find some some $c \in (a, b)$ for which

$$f(\vec{b}) - f(\vec{a}) \overset{\textbf{(a)}}{=} g'(c) \overset{\textbf{(b)}}{=} \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$$

where $\vec{c} = a + c \cdot (\vec{b} - \vec{a})$. As argued in (**1.**), this vector should successfully land on L. $\boxed{\xi}$

Exericse 4.[§] *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by* $f(0,0) = 0$ *and* $f(x,y) = \frac{xy(x^2-y^2)}{(x^2+y^2)}$ *if* $(x,y) \neq (0,0)$.

---

a) *Check if* f *is twice partially differentiable and if the second partial derivatives are continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$.

Check. First, let $(x,y) \neq (0,0)$. It should be clear that the numerator of f is a product of continuous functions, which is continuous. The only real obstacle is zeroes[¶] of the denominator, but those have already been excluded. The quotient is therefore continuous, and one may comfortably proceed to compute the first partial derivatives of f.

$$\begin{aligned}\partial_x f = \partial_x\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) &= \frac{(x^2+y^2)\cdot\partial_x\Big[xy(x^2-y^2)\Big] - xy(x^2-y^2)\cdot\partial_x\Big[x^2+y^2\Big]}{(x^2+y^2)^2} \\ &= \frac{(x^2+y^2)\cdot\Big[y(x^2-y^2)+xy(2x)\Big] - xy(x^2-y^2)\cdot\Big[2x\Big]}{(x^2+y^2)^2} \\ &= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}. \qquad \text{(a)} \\ \partial_y f = \partial_y\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) &= \frac{(x^2+y^2)\cdot\partial_y\Big[xy(x^2-y^2)\Big] - xy(x^2-y^2)\cdot\partial_y\Big[x^2+y^2\Big]}{(x^2+y^2)^2} \\ &= \frac{(x^2+y^2)\cdot\Big[x(x^2-y^2)+xy(-2y)\Big] - xy(x^2-y^2)\cdot\Big[2y\Big]}{(x^2+y^2)^2} \\ &= \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}. \qquad \text{(b)}\end{aligned}$$

By the same argument, we see that the numerators of $\partial_x f, \partial_y f$ are continuous, and that zeroes of their denominator are excluded. Therefore, both partials exist and are continuous. Finally, while one may proceed to compute the second−partial derivatives then check for continuity, it is sufficient to note that

1. $\partial_x f, \partial_y f$ are quotients of polynomials, which are differentiable functions,
2. the denominator $(x^2+y^2)^2$ of both partials *does not vanish*[‖] on $\mathbb{R}^2\setminus\{(0,0)\}$.

Then one concludes that all second−partial deirvatives exist and are continuous on $\mathbb{R}^2\setminus\{(0,0)\}$. $\boxed{\xi}$

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b) *Show that* f *is twice partially differentiable at* $(0,0)$.

[§] The exercise had mistakenly defined f as $f(x,y) = \frac{xy}{(x^2-y^2)(x^2+y^2)}$. This is problematic, since the given function is not twice partially differentiable at $(0,0)$. We apologise for this inconvenience.

[¶] The zeroes of a function g are inputs (x,y) satisfying $g(x,y) = 0$.

[‖] is never zero

Definition 2.7 (Partial Derivative).[**] Define $f : \mathcal{U} \to \mathbb{R}$ on an open subset $\mathcal{U} \subseteq \mathbb{R}^d$. The limit

$$\frac{\partial f}{\partial x_j} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_d) - f(x_1, \ldots, x_d)}{h}$$

if exists, is the partial derivative of f with respect to x_j. This is effectively taking the limit in one variable, that is x_j, with all other variables held constant.

Proof. To compute the partial derivatives at $(0,0)$, some care is required. For this, we utilise the limit definition of the partial derivative.

$$\partial_x f(0,0) := \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}, \qquad \partial_y f(0,0,) := \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}.$$

Next, $f(h,0) = f(0,0) = 0$ gives us that both partials exist and evaluate to zero. The second-order partial derivatives give

$$\partial_x^2 f(0,0) := \lim_{h\to 0} \left(\frac{\partial_x f(h,0) - \cancelto{0}{\partial_x f(0,0)}}{h} = \frac{\partial_x f(h,0)}{h} \right) \quad \partial_y \partial_x f(0,0) := \lim_{h\to 0} \left(\frac{\partial_x f(0,h) - \cancelto{0}{\partial_x f(0,0)}}{h} = \frac{\partial_x f(0,h)}{h} \right)$$

$$\partial_x \partial_y f(0,0) := \lim_{h\to 0} \left(\frac{\partial_y f(h,0) - \cancelto{0}{\partial_y f(0,0)}}{h} = \frac{\partial_y f(h,0)}{h} \right) \quad \partial_y^2 f(0,0) := \lim_{h\to 0} \left(\frac{\partial_y f(0,h) - \cancelto{0}{\partial_y f(0,0)}}{h} = \frac{\partial_y f(0,h)}{h} \right)$$

by definition. This reduces the task to computing the four numerators above. Since both $(h,0)$ and $(0,h)$ are non-zero, we may utilise the computations (a), (b) to get

$$\partial_x f(h,0) = \frac{\cancelto{0}{h^4 \cdot 0} + \cancelto{0}{4h^2 \cdot 0^3} - \cancelto{0}{0^5}}{(h^2+0^2)^2} = 0 \qquad \partial_x f(0,h) = \frac{\cancelto{0}{0^4 \cdot h} + \cancelto{0}{4 \cdot 0^2 \cdot h^3} - h^5}{(0^2+h^2)^2} = -h$$

$$\partial_y f(h,0) = \frac{h^5 - \cancelto{0}{4 \cdot h^3 \cdot 0^2} - \cancelto{0}{h \cdot 0^4}}{(h^2+0^2)^2} = h \qquad \partial_y f(0,h) = \frac{\cancelto{0}{0^5} - \cancelto{0}{4 \cdot 0^3 h^2} - \cancelto{0}{0 \cdot h^4}}{(0^2+h^2)^2} = 0$$

Then

$$\partial_x^2 f(0,0) := \lim_{h\to 0} \cancelto{0}{\left(\frac{\partial_x f(h,0)}{h} \right)} = 0 \qquad \partial_y \partial_x f(0,0) := \lim_{h\to 0} \left(\frac{\overbrace{\partial_x f(0,h)}^{=-h}}{h} \right) = -1$$

$$\partial_x \partial_y f(0,0) := \lim_{h\to 0} \left(\frac{\overbrace{\partial_y f(h,0)}^{=h}}{h} \right) = 1 \qquad \partial_y^2 f(0,0) := \lim_{h\to 0} \cancelto{0}{\left(\frac{\partial_y f(0,h)}{h} \right)} = 0$$

all second-order partial derivatives exist, and f must be twice−partially differentiable at $(0,0)$. $\boxed{\xi}$

Remark. We find it indeed useful (and perhaps necessary) that one exposes themselves to the limit definition of the derivative.

[**]See Section 2.2.1, Lectures 10 & 11

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c) *Compute* $\frac{\partial^2 f(x,y)}{\partial x \partial y}$ *and* $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ *at* $(x, y) = (0, 0)$. *Are these findings in contradiction to Schwartz's theorem?*

Define $f : \mathcal{U} \to \mathbb{R}$ on an open subset $\mathcal{U} \subseteq \mathbb{R}^d$.

Theorem 2.14 (Clairaut–Schwartz)**.** *Let f be continuous on $\mathcal{U}$, and suppose that all its second-order partial derivatives exist and are* ***continuous*** *on $\mathcal{U}$. Then,*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

for all $i, j \in \{1, \dots, d\}$.

Proof. From **vi.4.b**, we see that the mixed derivatives

$$\partial_{xy}(0,0)f = 1 \neq -1 = \partial_{yx} f(0,0)$$

are not equal. This, however, is **not** in contradiction to Schwartz's theorem, for two reasons:

- All partial derivatives are discontinuous at $(x, y) = (0, 0)$, failing the continuity requirement for Schwartz's theorem to hold;
- if Schwartz's theorem were to be incorrect, it is highly unlikely[††] that we disprove it in this exercise sheet. $\boxed{\xi}$

[††] In fact, with probability zero. . .

Sheet Seven

Keywords

Critical Points, The Hessian Matrix, Extrema test, Extremas and Saddle Points, The Extreme Value Problem, Method of Lagrangian Multipliers.

Contents

Exercise 1 *Determine whether the critical points of the following functions are local maxima, local minima or saddle points.*

Critical Point Analysis For the sake of this discussion, assume $f : \mathcal{U} \to \mathbb{R}^d$ to be defined on an open subset $\mathcal{U}$ of $\mathbb{R}^d$, and that it is twice-differentiable. A point $\vec{a} \in \mathbb{R}^d$ for which $\vec{\nabla} f(\vec{a}) = 0$ is said to be a *critical point*. The following theorem is pivotal to our discussion.

Proposition 2.22 (Critical Point Test).* *If f has a local maximum or minimum at $\vec{a}$, then $\nabla f(\vec{a}) = \vec{0}$.*

It gives us a way to check whether such a $\vec{a}$ exists. If it does, then it will satisfy this criterion. The following step is to determine whether $\vec{a}$ is an extremum of f. Luckily, we have

Definition 2.20 (Hessian)**.** The Hessian of f at $\vec{a}$ is defined as

$$H = H_f(\vec{a}) := \begin{pmatrix} \partial_{x_1}^2 f(\vec{a}) & \partial_{x_1}\partial_{x_2} f(\vec{a}) & \dots & \partial_{x_1}\partial_{x_d} f(\vec{a}) \\ \partial_{x_2}\partial_{x_1} f(\vec{a}) & \partial_{x_2}^2 f(\vec{a}) & \dots & \partial_{x_2}\partial_{x_d} f(\vec{a}) \\ \vdots & & \ddots & \\ \partial_{x_d}\partial_{x_1} f(\vec{a}) & \dots & \dots & \partial_{x_d}^2 f(\vec{a}) \end{pmatrix}.$$

It stores all second-order partial derivatives of f at $\vec{a}$.

Theorem 2.23 (Extrema Test)**.** *Let $\vec{a} \in \mathcal{U}$ be a critical point for which $\nabla f(\vec{a}) = 0$. If $\lambda_1, \lambda_2, \dots, \lambda_d$ are the eigenvalues of $H = H_f(\vec{a})$, then*

- $\lambda_i > 0$ *for all* $i \implies \vec{a}$ *is a local minimum.*
- $\lambda_i < 0$ *for all* $i \implies \vec{a}$ *is a local maximum.*
- $\lambda_i > 0$ *and* $\lambda_j < 0$ *for some indices* $i, j \implies \vec{a}$ *is a saddle point.*

The test is inconclusive otherwise.

To find local extrema of f, the recipe therefore goes as follows:

1. find all critical points of f using $\vec{\nabla} \cdot f(\vec{a}) = 0$;
2. compute $H_f(\vec{a})$ for all critical points $\vec{a}$;
3. compute the eigenvalues of $H_f(\vec{a})$;
4. utilise (**Theorem** 2.23) to reach a conclusion on $\vec{a}$.

With this, let us now start discussing the problems.

*Seek Page 2, Lectures 14 & 15.

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f. $f:\mathbb{R}^2\to\mathbb{R}$ *given by* $f(x,y)=x^3-3x+y^2$.

Solution. To obtain all critical points $\vec{a}$, we set $\vec{\nabla}f(\vec{x})=\tilde{\mathbf{0}}$,

$$\begin{aligned}(3x^2-3,2y)=(\partial_x f,\partial_y f)=:\vec{\nabla}f(\vec{x})=(0,0) &\iff x\in\{-1,1\} \text{ and } y=0\\ &\iff \vec{a}\in\{(-1,0),(1,0)\}\end{aligned}$$

With the initial computation $(\partial_x f,\partial_y f)=(3x^2-3,2y)$ in hand, we proceed with computing the Hessian (and consequently the characteristic equation).

$$\begin{aligned}H:=H_f(x,y):=\begin{pmatrix}\partial_x(\partial_x f) & \partial_x(\partial_y f)\\ \partial_y(\partial_x f) & \partial_y(\partial_y f)\end{pmatrix}=\begin{pmatrix}6x & 0\\ 0 & 2\end{pmatrix} &\implies \det(H-\lambda\cdot I)=(6x-\lambda)(2-\lambda)\\ &\implies \lambda\in\underbrace{\{6x,2\}}_{:=\Lambda(x,y)} \text{ for } \det(H-\lambda\cdot I)=0.\end{aligned}$$

To find the eigenvalues, one simply plugs-in $\vec{a}$ into the set $\Lambda(x,y)$ of eigenvalues, to get that

$$\begin{aligned}\Lambda(-1,0)=\{-6,2\} &\implies (-1,0) \text{ is a saddle;}\\ \Lambda(1,0)=\{6,2\} &\implies (1,0) \text{ is a local minimum;}\end{aligned}$$

by (**2.23**). $\boxed{\xi}$

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g. $g:\mathbb{R}^2\to\mathbb{R}$ *given by* $g(x,y)=x^4-y^6$.

Solution. Let us find the critical points of g.

$$(4x^3,-6y^5)=(\partial_x g,\partial_y g)=:\vec{\nabla}g(\vec{x})=(0,0)\iff \vec{a}=(0,0).$$

Computing the Hessian gives

$$\begin{aligned}H:=H_g(x,y):=\begin{pmatrix}\partial_x(\partial_x f) & \partial_x(\partial_y f)\\ \partial_y(\partial_x f) & \partial_y(\partial_y f)\end{pmatrix} &\implies H=\begin{pmatrix}12x^2 & 0\\ 0 & -30y^4\end{pmatrix}\\ \implies \det(H-\lambda\cdot I)=(12x^2-\lambda)(30y^4-\lambda) &\implies \lambda\in\underbrace{\{12x^2,-30y^4\}}_{:=\Lambda(x,y)} \text{ for } \det(H-\lambda\cdot I)=0.\end{aligned}$$

Then, plugging-in $\vec{a}$ into the set of eigenvalues, $\Lambda(0,0)=\{0\}$ implies that the test is inconclusive. Note, however, that while g does increase as we move away from $(0,0)$ along the $x-$axis, g decreases as we move away along the $y-$axis. With this, we may conclude that $(0,0)$ is neither a maximum nor a minimum, and must therefore be a saddle point. $\boxed{\xi}$

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h. *$h : \mathbb{R}^3 \to \mathbb{R}$ given by $h(x,y,z) = e^{-x^2-y^2} + z^3 - 3z$.*

Solution. Computing $\vec{\nabla} h(\vec{x})$ gives

$$\begin{aligned}(-2xe^{-x^2-y^2}, -2ye^{-x^2-y^2}, 3z^2-3) =: (\partial_x h, \partial_y h, \partial_z h) = 0 &\iff x = y = 0 \text{ and } z \in \{-1,1\} \\ &\iff \vec{a} \in \{(0,0,-1),(0,0,1)\}.\end{aligned}$$

The Hessian is therefore given by

$$H := H_h(x,y,z) := \begin{pmatrix} \partial_x(\partial_x f) & \partial_x(\partial_y f) & \partial_x(\partial_z f) \\ \partial_y(\partial_x f) & \partial_y(\partial_y f) & \partial_y(\partial_z f) \\ \partial_z(\partial_x f) & \partial_z(\partial_y f) & \partial_z(\partial_z f) \end{pmatrix} = \begin{pmatrix} e^{-x^2-y^2}\cdot(4x^2-2) & 4xy\cdot e^{-x^2-y^2} & 0 \\ 4xy\cdot e^{-x^2-y^2} & e^{-x^2-y^2}\cdot(4y^2-2) & 0 \\ 0 & 0 & 6z \end{pmatrix}$$

$$\begin{aligned}\implies H_h(0,0,z) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6z \end{pmatrix} \implies \det(H_h(0,0,z) - \lambda\cdot I) &= \begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & 6z-\lambda \end{vmatrix} \\ &= (2+\lambda)^2\cdot(6z-\lambda).\end{aligned}$$

The set of eigenvalues $\Lambda(0,0,z) = \{-2, 6z\}$ is obtained by setting the latter expression to zero. Then, computing $\Lambda(\vec{a})$ gives

$$\begin{aligned}\Lambda(0,0,1) = \{-2,6\} &\implies (0,0,1) \text{ is a saddle point} \\ \Lambda(0,0,-1) = \{-2,-6\} &\implies (0,0,-1) \text{ is a local maximum}\end{aligned}$$

by (**2.23**). $\boxed{\xi}$

Exercise 2 *Find the extreme values of $f(x,y) = x^3 - x + y^2 - 2y$ on the closed triangular region with vertices at $(-1,0), (1,0)$, and $(0,2)$.*

Theorem 2.24 (Extreme Value Theorem). *If f is continuous on a closed, bounsed set $S \subseteq \mathbb{R}^d$, then f attains a minimum or a maximum on S.*

A recipe to find extreme values on a closed, bounded set S is the following.[†]

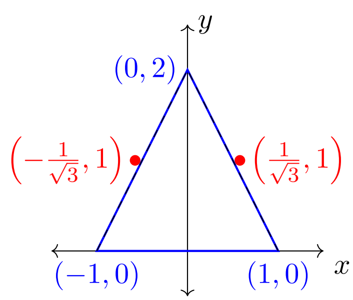
1. Find all critical points of f on S.
2. Find candidates for extreme values on the boundary of S.
3. Pick the smallest and largest values of f from (1), (2)

Solution. Let us attempt to find all critical points of f. Next, set

$$\vec{\nabla} f(\vec{a}) = 0 \iff \overbrace{(3x^2 - 1, 2y - 2)}^{(\partial_x f, \partial_y f)} = 0 \iff x = \pm\frac{1}{\sqrt{3}} \text{ and } y = 1$$
$$\iff \vec{a} \in \{(\frac{1}{\sqrt{3}}, 1), (\frac{-1}{\sqrt{3}}, 1)\}.$$

These points do not lie in the triangle.[‡] This, however, is not an issue. The recipe recommends that we resort to the triangle's boundary in search for extrema.

Critical Points on Δ



The idea is to *parameterise* f by the equations of each side. Every blue point belongs to either

1. $x \in [-1,1],\ y_1 = 0$ *base*
2. $x \in [-1,0],\ y_2 = 2x + 2$ *left-hand-side*
3. $x \in [0,1],\ y_3 = -2x + 2$ *right-hand-side*

This in turn induces three forms of $f(x,y) = x^3 - x + y^2 - 2y$, one for each possible parameterisation.

[†]Page 8, Lectures 14 & 15.

[‡]To see this, notice that the line connecting $(1,0), (0,2)$ has slope -2 with $y-$intercept 2, giving $y = -2x + 2$. All points above this line satisfy $y > -2x + 2$, which is true in the case of $(\frac{1}{\sqrt{3}}, 1)$. The same may be said for $(\frac{-1}{\sqrt{3}}, 1)$ by the symmetric setup.

$$f_1(x) := f(x)\Big|_{y=0} = x^3 - x + 0^2 - 2\cdot 0 = x^3 - x \tag{1}$$

$$f_2(x) := f(x)\Big|_{y=2x+2} = x^3 - x + (2x+2)^2 - 2(2x+2) = x^3 + 4x^2 + 3x \tag{2}$$

$$f_3(x) := f(x)\Big|_{y=-2x+2} = x^3 - x + (-2x+2)^2 - 2(-2x+2) = x^3 + 4x^2 - 5x. \tag{3}$$

The next step is to optimize in the variable x. This should be a simple exercise; one only needs to set the derivative to 0 to find the critical points.

$$\frac{d}{dx}\, f_1(x) = 0 \iff 3x^2 - 1 = 0 \iff x \in \left\{\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\} \qquad (x \in [-1,1])$$

$$\frac{d}{dx}\, f_2(x) = 0 \iff 3x^2 + 8x + 3 = 0 \iff x \in \left\{\frac{-4-\sqrt{7}}{3}, \frac{-4+\sqrt{7}}{3}\right\} \qquad (x \in [-1,0])$$

$$\frac{d}{dx}\, f_3(x) = 0 \iff 3x^2 + 8x - 5 = 0 \iff x \in \left\{\frac{-4-\sqrt{31}}{3}, \frac{-4+\sqrt{31}}{3}\right\} \qquad (x \in [0,1])$$

We eliminate $x = \frac{-4+\sqrt{7}}{3}, \frac{-4-\sqrt{31}}{3}$ as they do not fall within their respective intervals. One is then left with the following,

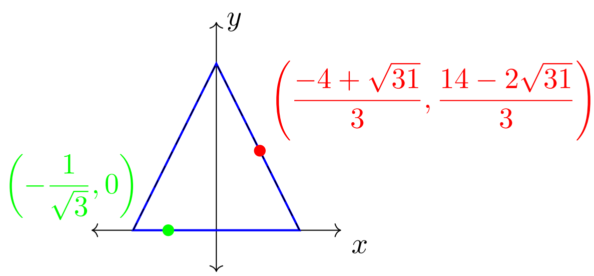
$$C = \left\{ \left(\frac{-1}{\sqrt{3}}, 0\right), \left(\frac{1}{\sqrt{3}}, 0\right), \left(\frac{-4+\sqrt{7}}{3}, \overbrace{2\cdot\frac{-4+\sqrt{7}}{3} - 2}^{:=y_2}\right), \left(\frac{-4+\sqrt{31}}{3}, \underbrace{-2\cdot\frac{-4+\sqrt{31}}{3} + 2}_{:=y_3}\right) \right\}.$$

These are all the critical pairs (x, y). Let us not forget to include the endpoints of the triangle,

$$C = \left\{ \left(\frac{-1}{\sqrt{3}}, 0\right), \left(\frac{1}{\sqrt{3}}, 0\right), \left(\frac{-4+\sqrt{7}}{3}, \frac{-14+2\sqrt{7}}{3}\right), \left(\frac{-4+\sqrt{31}}{3}, \frac{14-2\sqrt{31}}{3}\right), \overbrace{(-1,0), (1,0), (0,2)}^{\text{Triangle Endpoints}} \right\}.$$

In evaluating $f(\vec{c})$ for $\vec{c} \in C$, we find that $f\left(\frac{-4+\sqrt{31}}{3}, \frac{14-2\sqrt{31}}{3}\right) = \frac{308-62\sqrt{31}}{27}$ is an absolute minimum, whereas $f\left(\frac{-1}{\sqrt{3}}, 0\right) = \frac{2\sqrt{3}}{9}$ is an absolute maximum on the closed, bounded triangle. $\boxed{\xi}$

Maxima, Minima of f on Δ



Exercise 3 *Use the Lagrange multiplier method to solve Exercise 1 and Exercise 2 of the lab sheet* ***Optimization problems 1****.*

The Lagrangian Method is concerned with finding an extremum of $f(x,y) \in C^1$ given a constraint $g(x,y) = 0$. It formalises a key idea from (**v.4**), namely that the gradients of f, g must be parallel to obtain the direction of steepest change. Formally, if $\vec{a}$ is an extremum, then there exists a scalar λ for which

$$\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a}).$$

Definition (Lagrangian)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable. The Lagrangian of f is defined by

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda \cdot g(x,y)$$

where λ is the Lagrange multiplier, and $g(x,y)$ is a constraint function.

Then, one only needs to set $\vec{\nabla}\mathcal{L}(x,y,\lambda) = 0$ to

1. *enforce the parallel gradient condition,* $\vec{\nabla}\mathcal{L} := \vec{\nabla} f - \lambda \cdot \vec{\nabla} g = 0 \iff \vec{\nabla} f = \lambda \cdot \vec{\nabla} g$
2. *satisfy the constraint* $g(x,y) = 0$, since $\partial_\lambda \mathcal{L} := -g(x,y) = 0 \iff g(x,y) = 0$.

The Lagrangian naturally encodes all the information we require of the extreme value problem, and simplifies it to a single critical point analysis problem.[§]

---

1. *A farmer has 100 meter of fencing to enclose a rectangular garden on three sides (one side is along a river and does not need a fence). What are the best dimensions to maximize the area of the garden?*

Solution. The task is to maximize the area $A(x,y) = xy$ with the constraint $x + 2y = 100$. The factor of 1 next to x accounts for the missing side along the river. This setup induces a Lagrangian given by

$$\mathcal{L}(x,y,\lambda) = xy - \lambda(x + 2y - 100)$$

where the constraint is set to zero, and weighted by the Lagrangian multiplier λ. Proceed to compute critical points of $\mathcal{L}$,

$$\begin{aligned}
\overbrace{\vec{\nabla}\mathcal{L}(x,y,\lambda)}^{:=(\partial_x\mathcal{L},\partial_y\mathcal{L},\partial_\lambda\mathcal{L})} = (0,0,0) &\iff (y-\lambda, x-2\lambda, -x-2y+100) = (0,0,0) \\
&\iff y = \lambda, x = 2\lambda, x + 2y = 100 \\
&\iff 4\lambda = 100 \iff \lambda = 25 \\
&\iff x = 50, y = 25.
\end{aligned}$$

The dimensions are thus 50m × 25m, with the 50−meter side placed parallel to the river. $\boxed{\xi}$

[§]As examples, please seek Pages 11-13 of Lectures 14 & 15. There you may find out more on the Lagrange Method, and more generally on the Extremum with Constraints problem.

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2. *A company manufactures cylindrical cans using a sheet of metal. The company must design a can that has a fixed volume of V cubic centimeters. The cost of the metal required to make the can depends on the total surface area, as metal is used for the sides, top, and bottom of the can. Determine the values of the radius r and height h that minimize the amount of metal used, and therefore minimize the cost, while ensuring the can has the required volume.*

Solution. A cylindrical can has surface area $S(r,h) = 2\pi r^2 + 2\pi r \cdot h$. The task is to minimize $S(r,h)$, constrained with the volume $V = \pi r^2 h$ of the cylinder. The Lagrangian is therefore given by

$$\mathcal{L}(r,h,\lambda) = 2\pi r^2 + 2\pi r \cdot h - \lambda(V - \pi r^2 \cdot h)$$

The critical values are given by

$$\begin{aligned}\vec{\nabla}\mathcal{L}(r,h,\lambda) = (0,0,0) &\iff (4\pi r + 2\pi h - \lambda \cdot 2\pi rh, 2\pi r - \lambda \cdot \pi r^2, -V + \pi r^2 \cdot h) = (0,0,0) \\ &\iff \begin{cases} 4\pi r + 2\pi h = \lambda \cdot 2\pi rh & (a) \\ 2\pi r = \lambda \cdot \pi r^2 & (b) \\ V = \pi r^2 \cdot h & (c) \end{cases}\end{aligned}$$

We will solve this by substitution, starting with (b).

$$2\pi r = \lambda \cdot \pi r^2 \overset{\times\frac{1}{r\pi}}{\iff} 2 = \lambda \cdot r \overset{\times\frac{1}{r}}{\iff} \lambda = \frac{2}{r}. \tag{1}$$

Substituting in (a) gives

$$4\pi r + 2\pi h = \lambda \cdot 2\pi rh \overset{\times\frac{1}{2\pi}}{\iff} 2r + h = \lambda \cdot rh \overset{(1)}{\iff} 2r + h = \frac{2}{\not r} \cdot \not r h \overset{-h}{\iff} 2r = h. \tag{2}$$

Finally, (c) tells us that

$$V = \pi r^2 \cdot h \overset{(2)}{\iff} V = \pi r^2 \cdot (2r) \overset{\times\frac{1}{2\pi}}{\iff} \frac{V}{2\pi} = r^3 \overset{\sqrt[3]{(\cdot)}}{\iff} r = \sqrt[3]{\frac{V}{2\pi}}.$$

Given a fixed volume V, we set $r = \sqrt[3]{\frac{V}{2\pi}}$ and $h = 2r = \sqrt[3]{\frac{4V}{\pi}}$ to minimize the surface area. $\boxed{\xi}$

Exercise 4 *Consider the curve defined by the equation $x_1^2 + x_2^2 = 3$. Find the point on this curve that is closest to the point $(3, 0)$ by using the Lagrange multiplier method.*
Hint: Find first the function that describes the distance between points in the plane and the point $(3, 0)$.

Solution. The distance function in $\mathbb{R}^2$ has to be the Euclidean metric, given by

$$d_2(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}.$$

For $y = (3, 0)$, this would become

$$d_2(x, y) = \sqrt{(x_1 - 3)^2 + x_2^2} = \sqrt{x_1^2 + x_2^2 - 6x_1 + 9}.$$

Interlude. Things get even more interesting if we choose $x : x_1^2 + x_2^2 = 3$ on the curve, as the distance function simplifies to

$$d_2(x, y) = \sqrt{\underbrace{x_1^2 + x_2^2}_{:=3} - 6x_1 + 9} = \sqrt{6} \cdot \sqrt{2 - x_1}.$$

This is minimized upon maximizing x_1. With freedom to choose $x_1 \in [0, \sqrt{3}]$, set $x_1 = \sqrt{3}$ and $x_2 = 0$. Then $(\sqrt{3}, 0)$ is the closest point to $(3, 0)$. The exercise, however, asks us to specifically use the method of Lagrange multipliers, therefore we will not proceed in this fashion.

Minimizing $d_2(x, y)$ is equivalent to minimizing its square. Accordingly, describe the Lagrangian by

$$\mathcal{L}(x, y, \lambda) = x_1^2 + x_2^2 - 6x_1 + 9 - \lambda(x_1^2 + x_2^2 - 3).$$

Its critical points may be obtained as follows,

$$\begin{aligned}
\vec{\nabla}\mathcal{L}(x, y, \lambda) = (0, 0, 0) \iff (0, 0, 0) &= (2x_1 - 6 - 2\lambda \cdot x_1, 2x_2 - \lambda \cdot 2x_2, -x_1^2 - x_2^2 + 3) \\
&= (2x_1(1 - \lambda) - 6, 2x_2(1 - \lambda), x_1^2 + x_2^2 - 3)
\end{aligned}$$

$$\iff \begin{cases} 2x_1 \cdot (1 - \lambda) = 6 & (a) \\ x_2 \cdot (1 - \lambda) = 0 & (b) \\ x_1^2 + x_2^2 - 3 = 0 & (c) \end{cases} \quad \begin{aligned} &\overset{(*)}{\iff} (1 - \lambda) \neq 0 \\ &\overset{(b)}{\iff} x_2 = 0 \\ &\overset{(c)}{\iff} x_1 = \sqrt{3} \end{aligned}$$

The iff statement (*) is true, as $(1 - \lambda) = 0$ would imply that (a), (b) must both evaluate to zero, a contradiction. The result $x = (\sqrt{3}, 0)$ is consistent with that obtained during the earlier interlude. $\boxed{\xi}$

Sheet Eight

Keywords

Extreme Value Problems, Linear Programming, The Fundamental Theorem of Linear Programming, Graphical Solutions, Simplex Method.

Contents

Exercise 1 *Write the following problem in the standard form of linear programming.*

$$\begin{array}{c} \textit{Maximize} \\ Z = 2x_1 + x_2 - 3x_3 \\ \textit{subject to the constraints} \end{array} \quad \begin{cases} x_1 + x_2 \leq 40 \\ 4x_1 + x_2 \leq 100 \\ x_1 + x_2 + x_3 \geq 20 \\ x_1, x_2 \geq 0 \end{cases}$$

and no further restrictions on the sign of x_3.

Definition 2.26. Let $m, n \in \mathbb{N}$ with $n \geq m$. A linear programming problem is in *standard form* if written as

$$\text{Minimize } Z = c^T x \text{ subject to constraints } Ax = b \text{ and } x \geq 0.$$

Here, $x, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.*

Solution. We give a recipe† to bring any linear programming problem to a standard form.

1. *Convert to a minimization problem.* As per (2.26), we would like to turn our maximization problem into a minimization problem. This is simple, as

$$\text{Maximize } Z = 2x_1 + x_2 - 3x_3 \iff \text{Minimize } \tilde{Z} = -2x_1 - x_2 + 3x_3.$$

2. *Handle unrestricted variables.* Notice that x_3 can take positive or negative values. For this, write $x_3 = x_3^+ - x_3^-$ with $x_3^+, x_3^- \geq 0$. From now on, we minimize $\tilde{Z} = -2x_1 - x_2 + 3x_3^+ - 3x_3^-$ and replace x_3 with the introduced pair of variables.

3. *Introduce slack variables to turn inequality constraints into equality constraints.* For $\leq$, we would like to add a positive amount to make it an equality; similarly for $\geq$. This is best demonstrated by an example.

$$\begin{array}{c} \text{Minimize} \\ \tilde{Z} = -2x_1 - x_2 + 3x_3^+ - 3x_3^- \\ \text{subject to constraints} \end{array} \begin{cases} x_1 + x_2 \leq 40 & \iff x_1 + x_2 + s_1 = 40 \\ 4x_1 + x_2 \leq 100 & \iff 4x_1 + x_2 + s_2 = 100 \\ x_1 + x_2 + x_3^+ - x_3^- \geq 20 & \iff x_1 + x_2 + x_3^+ - x_3^- - s_3 = 20 \\ x_1, x_2 \geq 0 & \iff x_1, x_2, x_3^+, x_3^-, s_1, s_2, s_3 \geq 0 \end{cases}.$$ $\boxed{\xi}$

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*Page 4, Section 2.5, Lectures 16 & 17.
†Page 5, Section 2.5, Lectures 16 & 17.

Exercise 2

$$\begin{array}{c}\textit{Maximize}\\ Z = x_1 + 2x_2\\ \textit{subject to constraints}\end{array} \quad \begin{cases} x_1 + 3x_2 & \leq 8 \\ x_1 + x_2 & \leq 4 \\ x_1, x_2 & \geq 0 \end{cases}.$$

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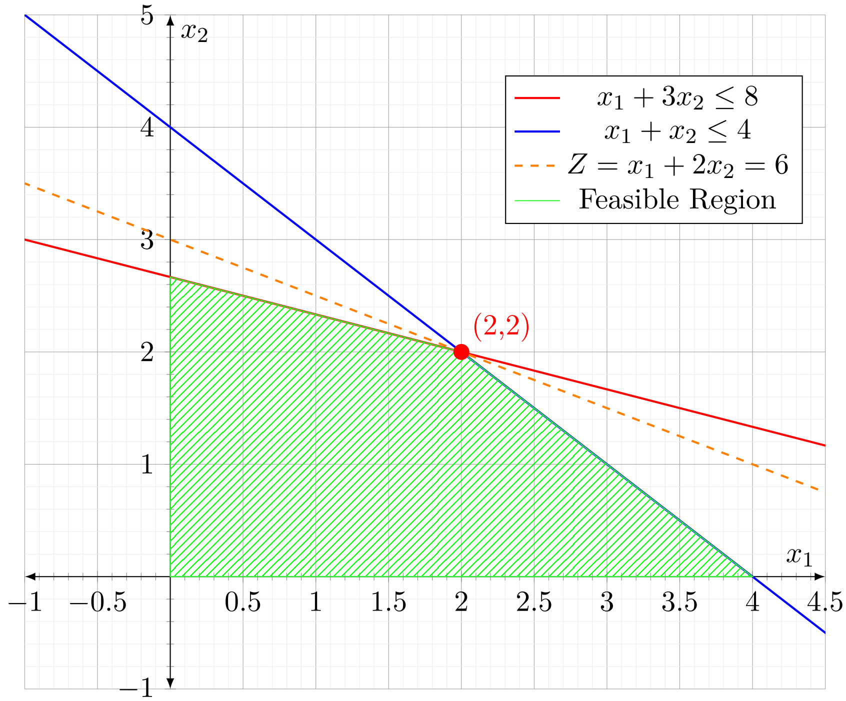
a) *Solve the problem graphically.*

Sketch.



ξ

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b) *Write the problem in the standard form of linear programming.*

Solution. As shown earlier in (**viii.1**), we apply the recipe.

1. *Convert into a minimization problem.* Maximize $Z = x_1 + 2x_2 \iff$ Minimize $\tilde{Z} = -x_1 - 2x_2$.
2. *Handle unrestricted variables.* This is not an issue here, all variables are restricted.
3. *Introduce slack variables.* This gives rise to the following.

$$\text{Minimize } \tilde{Z} = -x_1 - 2x_2 \text{ subject to constraints } \begin{cases} x_1 + 3x_2 + s_1 & = 8 \\ x_1 + x_2 + s_2 & = 4 \\ x_1, x_2, s_1, s_2 & \geq 0 \end{cases}.$$

ξ

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c) *Solve the problem with the simplex method.*

Solution. We start by initialising our simplex *tableau*. This is done using coefficients of the linear constrained system, written in standard form. Coefficients of the objective function $\tilde{Z} = -x_1 - x_2 - 0 \cdot s_1 - 0 \cdot s_2$ are entered in the lower-most row, with the right-most entry set to $\mathbf{0}$.

$$\begin{cases} x_1 + 3x_2 + s_1 & = 8 \\ x_1 + x_2 + s_2 & = 4 \\ x_1, x_2, s_1, s_2 & \geq 0 \end{cases} \iff$$

Basis	x_1	x_2	s_1	s_2	**RHS**
	1	3	1	0	8
	1	1	0	1	4
$\tilde{Z}$	-1	-2	0	0	$\mathbf{0}$

Given 4 variables and 2 constraints, one would like to choose 2 basic variables to form a **basis**. We choose s_1, s_2 for the basis, since $(x_1, x_2, s_1, s_2) = (0, 0, 8, 4)$ is a feasible[‡] solution. It is, however, suboptimal, and a most−suitable solution shall be obtained upon termination of the simplex algorithm.

1. **Entering Variable.** *Choose is the most-negative coefficient in row* $\tilde{Z}$.
   ↪ This is x_2 with value -2.

2. **Minimal Row.** *Choose the row with a minimum positive ratio between* ***RHS*** *and the coefficient in the entering variable column.*
   ↪ The first row has a ratio of $\frac{(\mathbf{RHS})_1}{(x_2)_1} = \boxed{8 \div 3}$, whereas the second row has one of $\frac{(\mathbf{RHS})_2}{(x_2)_2} = 4 \div 1$.

3. **Leaving Variable.** *In the basic solution matrix, choose the pivot variable in the minimal row.*
   ↪ The basic solution matrix is initialised with columns s_1, s_2. The variable s_1 holds the $\boxed{\text{pivot}}$ in the first row, and is thus leaving.

The basic solution matrix must always form a permutation of the identity matrix. A new $\boxed{\text{pivot}}$ is therefore initialised at the first entry of column x_2.

Basis	x_1	x_2	s_1	s_2	**RHS**
s_1	1	$\boxed{3}$	$\boxed{1}$	0	8
s_2	1	1	0	1	4
$\tilde{Z}$	-1	-2	0	0	$\mathbf{0}$

$\times\frac{1}{3}$, -1, 2 $\iff$

Basis	x_1	x_2	s_1	s_2	**RHS**
x_2	$\frac{1}{3}$	1	$\frac{1}{3}$	0	$\frac{8}{3}$
s_2	$\frac{2}{3}$	0	$-\frac{1}{3}$	1	$\frac{4}{3}$
$\tilde{Z}$	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	$\frac{16}{3}$

Notice that x_2 has entered for s_1 in the basis, in the sense that our *tableau* replaces s_1 with x_2. Let us apply the same iteration once again.

1. **Entering Variable.** The x_1−column has the only negative coefficient $(-\frac{1}{3})$.

2. **Minimal Row.** The second row has a ratio of $\frac{(\mathbf{RHS})_1}{(x_1)_1} = \frac{4}{3} \div \frac{2}{3} < \frac{(\mathbf{RHS})_2}{(x_1)_2} = \frac{8}{3} \div \frac{1}{3}$.

3. **Leaving Variable.** s_2 holds the pivot in the second row.

[‡] All variables are non-negative.

The next step is create a pivot in the second entry of column x_1.

Basis	x_1	x_2	s_1	s_2	RHS
x_2	$\frac{1}{3}$	1	$\frac{1}{3}$	0	$\frac{8}{3}$
s_2	$\boxed{\frac{2}{3}}$	0	$-\frac{1}{3}$	$\boxed{1}$	$\frac{4}{3}$
$\tilde{Z}$	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	$\frac{16}{3}$

$\times\frac{-1}{2}$ $\quad \mid \times \frac{3}{2}$ $\quad \times\frac{1}{2}$ $\quad \Longleftrightarrow$

Basis	x_1	x_2	s_1	s_2	RHS
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2
x_1	1	0	$-\frac{1}{2}$	$\frac{3}{2}$	2
$\tilde{Z}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	6

The algorithm terminates when there are no more negative entries in the $\tilde{Z}-$row. We may stop and read out the solution.

- *The bottom right corner gives the negative of the objective function.* This tells us that the objective function $\tilde{Z}$ attains a minimum of -6. Therefore, $Z = -\tilde{Z}$ attains its maximum at 6.
- *The values for the basic variables can be can be read off the right-hand column.* One may quickly verify that $x_2 = x_1 = 2$ give the maximum $Z = 6$.

Most conveniently, this is consistent with the graphical solution we obtained earlier in **a)**. $\boxed{\xi}$

The Simplex Construction Let us give some insight into how the simplex algorithm works.

Theorem 2.27 (Fundamental Theorem of Linear Programming)**.** *If a linear programming problem (in standard form) has optimal solutions, then there is an optimal solution on the corner of the feasible region.*

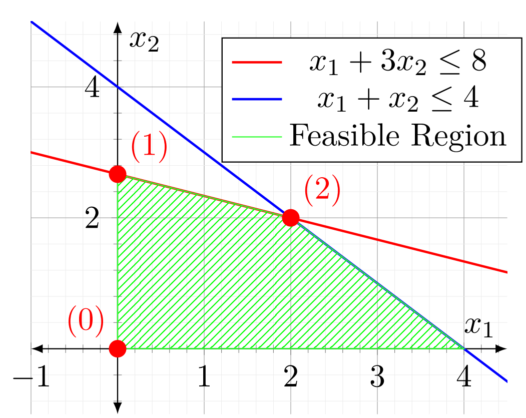
Geometrically, the simplex method *jumps* from one corner to the other. We had three iterations,

(0) $(x_1, x_2, s_1, s_2) = (0, 0, 8, 4)$
$\hookrightarrow (x_1, x_2) = (0, 0)$;

(1) $(x_1, x_2, s_1, s_2) = (\frac{8}{3}, 0, \frac{4}{3}, 0)$
$\hookrightarrow (x_1, x_2) = (0, \frac{8}{3})$;

(2) $(x_1, x_2, s_1, s_2) = (2, 2, 0, 0)$
$\hookrightarrow (x_1, x_2) = (2, 2)$.

The Simplex Method



In the step $(0) \mapsto (1)$, we chose x_2 to enter and s_1 to leave. In the first constraint $\cancel{x_1}^{0} + 3x_2 + s_1 = 8$, we set $s_1 = 0$ and solve $x_2 \rightsquigarrow \frac{8}{3}$. This gives the next corner point $(x_1, x_2) = (0, \frac{8}{3})$.

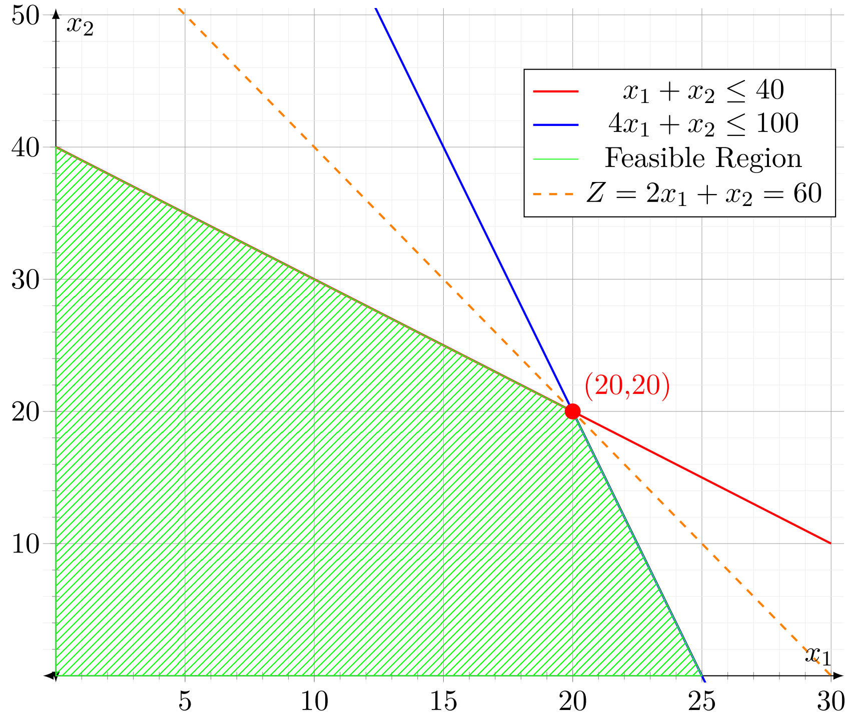
The choice of variables corresponds to the direction of steepest increase, whereas the solution geometrically corresponds to moving along that direction until one hits the next corner point. The simplex construction re-iterates this loop until an optimal solution is reached.

Exercise 3 *Maximize $Z = 2x_1 + x_2$ subject to the constraints* $\begin{cases} x_1 + x_2 & \leq 40 \\ 4x_1 + x_2 & \leq 100 \\ x_1, x_2 & \geq 0 \end{cases}$.

---

a) *Solve the problem graphically.*

Sketch.



$\boxed{\xi}$

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b) *Write the problem in the standard form of linear programming.*

Solution. Apply the recipe shown in **Exercise 1**.

1. *Convert into a minimization problem.* Maximize $Z = 2x_1 + x_2 \iff$ Minimize $\tilde{Z} = -2x_1 - x_2$.
2. *Handle unrestricted variables.* This is not an issue here, all variables are restricted.
3. *Introduce slack variables.* This gives rise to the following.

$$\text{Minimize } \tilde{Z} = -2x_1 - x_2 \text{ subject to constraints } \begin{cases} x_1 + x_2 + s_1 & = 40 \\ 4x_1 + x_2 + s_2 & = 100 \\ x_1, x_2, s_1, s_2 & \geq 0 \end{cases}.$$

$\boxed{\xi}$

c) *Solve the problem with the simplex method.*

Solution. We proceed in the exact same manner as in (**viii.3.c**). Let us initialise the simplex *tableau* with

$$\begin{cases} x_1 + x_2 + s_1 &= 40 \\ 4x_1 + x_2 + s_2 &= 100 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{cases} \iff$$

Basis	x_1	x_2	s_1	s_2	**RHS**
s_1	1	1	1	0	40
s_2	4	1	0	1	100
$\tilde{Z}$	-2	-1	0	0	**0**

With the choice of s_1, s_2 as basic variables, we are ready to start with optimization.

1. **Entering Variable.** Column x_1 has the most negative entry (-2) in row $\tilde{Z}$.
2. **Minimal Row.** The second row has a smaller ratio $\frac{(\mathbf{RHS})_2}{(x_1)_2} = \frac{100}{4}$ compared to $\frac{(\mathbf{RHS})_1}{(x_1)_1} = \frac{40}{1}$.
3. **Leaving Variable.** s_2 holds the pivot in the second row of the basic solution matrix.

Basis	x_1	x_2	s_1	s_2	**RHS**
s_1	1	1	1	0	40
s_2	4	1	0	1	100
$\tilde{Z}$	-2	-1	0	0	**0**

$\times \frac{-1}{4}$ $\quad | \times \frac{1}{4}$ $\quad \times \frac{1}{2}$ $\quad \iff$

Basis	x_1	x_2	s_1	s_2	**RHS**
s_1	0	$\frac{3}{4}$	1	$-\frac{1}{4}$	15
x_1	1	$\frac{1}{4}$	0	$\frac{1}{4}$	25
$\tilde{Z}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	50

The second iteration gives

1. **Entering Variable.** Column x_2 has the only negative entry ($-\frac{1}{2}$) in row $\tilde{Z}$.
2. **Minimal Row.** The first row has a ratio $\frac{(\mathbf{RHS})_1}{(x_2)_1} = 15 \div \frac{3}{4} < 25 \div \frac{1}{4} = \frac{(\mathbf{RHS})_2}{(x_2)_2}$.
3. **Leaving Variable.** s_1 holds the pivot in the first row of the basic solution matrix.

Basis	x_1	x_2	s_1	s_2	**RHS**
s_1	0	$\frac{3}{4}$	1	$-\frac{1}{4}$	15
x_1	1	$\frac{1}{4}$	0	$\frac{1}{4}$	25
$\tilde{Z}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	50

$-\frac{1}{3}$ $\quad | \times \frac{4}{3}$ $\quad \frac{2}{3}$ $\quad \iff$

Basis	x_1	x_2	s_1	s_2	**RHS**
x_2	0	1	$\frac{4}{3}$	$-\frac{1}{3}$	20
x_1	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	20
$\tilde{Z}$	0	0	$\frac{2}{3}$	$\frac{1}{3}$	60

and we may now read off the solution.

- The objective function $\tilde{Z}$ attains a minimum of minus the lower-right entry, i.e. $\tilde{Z} = -60$. This implies that $Z = 60$ is the solution to the maximization problem.
- One may verify that the maximum $Z = 60$ is attained at $x_1 = x_2 = 20$.

This is once again consistent with the graphical result. $\boxed{\xi}$