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Mathematical Modeling
Solution Sheet

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Abstract

This document outlines detailed solutions to homework assignments for Mathematical Modeling, taught by Professor Nikolai Leopold in the Spring of 2025. Written in an expository style, the goal is to familiarise and equip the reader with the right ideas for a more developed treatment of the subject.

[1] The author wishes to acknowledge and thank Professor Nikolai Leopold for his valuable remarks and comments.

Contents

Sheet One

Keywords

Newton's Law of Cooling, Separation of Variables, Initial Conditions, Linear Inhomogeneous Differential Equations.

Contents

Exericse 1. *A cup of tea at 90° C is in a room at constant temperature of 20° C. By Newton's Law of Cooling, the change of the temperature in time is proportional to the difference between the current temperature of the tea and the room temperature. It is not affected by the amount of tea.*

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a) *Derive a differential equation that models the temperature $T(t)$ over time. Afterwards, find the solution of the differential equation.*

Proof. Let $k > 0$ and T be the temperature of tea at time t, with initial temperature T_0. The following differential equation

$$\frac{d}{dt}\, T = -k(T - 20)$$

expresses the assumption that the change in temperature is proportional to the difference of T and 20° C. By separation of variables, we obtain

$$\begin{aligned}\frac{dT}{T-20} = -kdt \implies & \int_{T_0}^{T} \frac{d\tilde{T}}{\tilde{T}-20} = \int_0^t -k\ d\tilde{t} \\ \implies & \ln(\tilde{T}-20)\Big|_{T_0}^{T} := \ln(T-20) - \ln(T_0 - 20) = -kt \\ \overset{e^{(\cdot)}}{\implies} & T - 20 = e^{-kt}(T_0 - 20) \implies T = 20 + e^{-kt}(T_0 - 20)\end{aligned}$$

For an initial temperature of 90° C, we get that $T(t) = 20 + e^{-kt} \cdot 70$ solves the differential equation. $\boxed{\xi}$

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b) *The temperature of the tea is 70° C after 5 minutes. Determine the constant which describes the speed of cooling. When will the temperature of the tea be 40° C?*

Solution. It is given that $T(5) = 70$, so that

$$70 = T(5) := 20 + 70e^{-k\cdot 5} \implies e^{-k\cdot 5} = \frac{5}{7} \implies -5k = \ln(\frac{5}{7}) \implies k = -\frac{\ln(\frac{5}{7})}{5}$$

gives the cooling rate. For $T(t) = 40$, this is just

$$\begin{aligned}40 = T(t_{40}) = 20 + 70e^{-k\cdot t_{40}} \implies & \frac{20}{70} = e^{-k\cdot t_{40}} \\ \implies & \ln(\frac{2}{7}) = -k \cdot t_{40} \\ \implies & -\frac{\ln(\frac{2}{7})}{k} := \cancel{-}\frac{\ln(\frac{2}{7})}{\cancel{-}\frac{\ln(\frac{5}{7})}{5}} = t_{40} \\ \implies & t_{40} = 5 \cdot \frac{\ln(\frac{2}{7})}{\ln(\frac{5}{7})} \approx 18.62 \text{ minutes.}\end{aligned}$$

$\boxed{\xi}$

Exercise 2. *Find the solution of the differential equation* $\frac{d}{dx}y(x) = 2xy(x) + x^3$.

Solution. Note that the aforementioned differential equation is one that is *linear* and *inhomogeneous*. We therefore make use of the Ansatz $y = uv$ to get

$$u'v + uv' =: \frac{d}{dx}\underbrace{(uv)}_{:=y} = 2x \cdot uv + x^3. \tag{*}$$

Next, notice that the choices

$$u' = 2x \cdot u \qquad \& \qquad v' = \frac{x^3}{u}, \tag{i.1}$$

satisfy (*) by design. We equated the two sides of (*) by comparison. This is the core idea, and with that we may proceed to solve two simpler differential equations, starting with u.

$$u' = 2x \cdot u \implies \int_{u_0}^{u} \frac{du}{u} = \int_{x_0}^{x} 2x \; dx \implies \underbrace{\ln(u) - \ln x_0}_{\ln(\frac{u}{u_0})} = x^2 - x_0^2$$

$$\implies u = u_0 e^{x^2 - x_0}$$

Then, v' may be written as

$$\frac{d}{dx}v = u_0^{-1}x^3 e^{(x_0^2 - x^2)} \implies v = v_0 + \underbrace{u_0^{-1}e^{x_0^2}}_{\textbf{constants}} \cdot \int_{x_0}^{x} x^3 e^{-x^2} dx$$

with the integral evaluating to

$$\int_{x_0}^{x} x^3 e^{-x^2} dx \underset{dt=2xdx}{\overset{t=x^2}{=}} \frac{1}{2}\int_{x_0^2}^{x^2} te^{-t}dt = \frac{1}{2}\left(-te^{-t} - \overbrace{\int -e^{-t}\; dt}^{:=e^{-t}}\right)\Bigg|_{x_0^2}^{x^2} = -\frac{1}{2}e^{-t}(t+1)\Big|_{x_0^2}^{x^2}$$

$$= \frac{1}{2}\left(e^{-x_0^2}(x_0^2+1) - e^{-x^2}(x^2+1)\right).$$

Therefore we get

$$\begin{aligned} v &= v_0 + \frac{1}{2}u_0^{-1}e^{x_0^2}\left(\cancelto{1}{e^{-x_0^2}}(x_0^2+1) - e^{-x^2}(x^2+1)\right) \\ &= v_0 + \frac{1}{2}u_0^{-1}\left(x_0^2 + 1 - e^{x_0^2 - x^2}(x^2+1)\right) \end{aligned}$$

Finally, recall that $y = uv$ gives the solution.

$$\begin{aligned} y &= u_0 \cdot e^{x^2 - x_0^2}\left(v_0 + \frac{1}{2}u_0^{-1}\left(x_0^2 + 1 - e^{x_0^2 - x^2}(x^2+1)\right)\right) \\ &= \underbrace{u_0 v_0}_{:=y_0} e^{x^2 - x_0^2} + \frac{1}{2}e^{x^2 - x_0}(x_0^2+1) - \frac{1}{2}(x^2+1). \end{aligned}$$

$\boxed{\xi}$

Remark. For more on this technique, seek page 5 of lectures 2, 3.

Exercise 3. *Find the solution of the differential equation* $\frac{d}{dx}y(x) = 2xy(x) + (1+x^2)y^2(x)$.

Proof. Using the *ansatz* $y = uv$ we get that

$$\frac{d}{dx}\, uv = u'v + uv' = 2x \cdot uv + (1+x^2)\cdot u^2v^2$$

By comparison of terms, set

$$u' = 2xu \tag{*}$$
$$v' = (1+x^2)\cdot uv^2 \tag{**}$$

then clearly $u = C_u \cdot e^{x^2}$. The second equation gives

$$\begin{aligned}\frac{dv}{dx} =: v' = (1+x^2)\cdot C_u \cdot e^{x^2} \cdot v^2 \implies & \frac{1}{v^2}\, dv = C_u \cdot \left(e^{x^2} + x^2 e^{x^2}\right)\cdot dx \\ \implies & -\frac{1}{v} = C_u \cdot \int (1+x^2)\cdot e^{x^2} dx \\ \implies & v = -C_u^{-1} \cdot \frac{1}{\int (1+x^2)\cdot e^{x^2} dx}.\end{aligned}$$

All-in-all, this gives

$$y = \cancel{-C_u^{-1}} \cdot \frac{\cancel{C_u} \cdot e^{x^2}}{\int (1+x^2)\cdot e^{x^2} dx} = -\frac{e^{x^2}}{\int (1+x^2)\cdot e^{x^2} dx}.$$

We do not attempt to compute the integral $\int (1+x^2)\cdot e^{x^2} dx$, since it is non-elementary. This concludes the argument. $\boxed{\xi}$

Sheet Two

Keywords

Linear Homogeneous Differential Equations, Linear Inhomogeneous Differential Equations, the Logistic Equation, Vector Fields, Systems of Differential Equations.

Contents

Exercise 1 *Consider the second-order inhomogeneous differential equation*

$$\frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x) = e^x.$$

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a) *Find the general solution to the corresponding homogeneous equation.*

Solution. A general solution to the homogeneous equation of the form

$$a \cdot \frac{d^2}{d^2x}y(x) + b \cdot \frac{d}{dx}y(x) + c \cdot y(x) = 0$$

is well-studied[1], and the idea is to consider the choice of $y = e^{\lambda x}$. This is great, since this choice

$$\begin{cases} y(x) = e^{\lambda x} \\ \frac{d}{dx}y(x) = \lambda e^{\lambda x} \\ \frac{d^2}{d^2x}y(x) = \lambda^2 e^{\lambda x} \end{cases} \implies a \cdot \lambda^2 e^{\lambda x} + b \cdot \lambda e^{\lambda x} + c \cdot e^{\lambda x} = e^{\lambda x}\left(a \cdot \lambda^2 + b \cdot \lambda + c\right) = 0$$

$$\implies a \cdot \lambda^2 + b \cdot \lambda + c = 0 \qquad \text{(since } e^{\lambda x} > 0\text{)}$$

in turn yields a polynomial equation, which we are very happy to solve. It simplifies the task to finding λ. Let us indeed proceed in that very-same spirit, and solve for $a = 1, b = -2, c = 2$.

$$1 \cdot \lambda^2 - 3 \cdot \lambda + 2 = 0 \implies \lambda_1 = 1, \lambda_2 = 2 \implies \begin{cases} y_1(x) = e^x \\ y_2(x) = e^{2x} \end{cases}.$$

Notice that we ended up with two solutions, when we were looking for one. This is a good moment to recall that any linear combination of homogeneous solutions gives a homogeneous solution,

$$\boxed{y(x) = Ae^x + Be^{2x} \qquad \text{for } A, B \in \mathbb{R}}$$

by linearity of the differential operator. As an exercise, try to verify yourself that

$$\frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x) = 0$$

for the choices of

- $y(x) = e^x$
- $y(x) = Ae^x$
- $y(x) = e^x + e^{2x}$
- $y(x) = Ae^x + Be^{2x}$.

$\boxed{\xi}$

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[1] see page 10, lectures 2 and 3 on moodle.

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b) *Consider the second-order inhomogeneous differential equation*

$$\frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x) = e^x.$$

Find the general solution to the inhomogeneous equation.

Proof. We will use the following lemma from class,

Lemma 1.5 (Lectures 2,3 - Page 9)**.** *Consider the following inhomogeneous equation*

$$a \cdot \frac{d^2}{d^2x}y(x) + b \cdot \frac{d}{dx}y(x) + c \cdot y(x) = f(x).$$

The general solution $y(x)$ may be written as

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the general solution of the homogeneous equation, and $y_p(x)$ is a particular solution of the inhomogeneuos equation.

In the previous problem, we established the homogeneous solution to be

$$y_h(x) = Ae^x + Be^{2x} \qquad \text{for } A, B \in \mathbb{R}.$$

If we can find a particular solution $y_p(x)$, then we are done. Now, to find $y_p(x)$, one may proceed in the spirit of *the quick method*[2] of undetermined coefficients. One attempts to guess an *Ansatz* of a similar structure to $f(x)$, equal to e^x in our case. Another way to proceed is with the *variation of constants*, which is more informative.[3]

The idea is to equate $y_p(x)$ to $y_h(x)$, but with coefficients that vary in x. Instead of constants A, B, we write

$$y_p(x) = A(x)e^x + B(x)e^{2x}$$

as functions of x. This reduces the problem to that of finding the coefficients $A(x), B(x)$, since that automatically describes $y_p(x)$. With this description, let us compute $\frac{d}{dx}y_p(x), \frac{d^2}{d^2x}y_p(x)$.

$$\frac{d}{dx}y_p(x) := A'(x)e^x + B'(x)e^{2x} + A(x)e^x + 2 \cdot B(x)e^{2x}.$$

Let us enforce a restriction on $A'(x), B'(x)$. This will become very useful in a moment.

$$A'(x)e^x + B'(x)e^{2x} = 0. \tag{1}$$

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[2]Page 2, Lectures 4,5, on moodle.
[3]This is a subjective opinion.

This implies

$$\frac{d}{dx}y_p(x) =^{(1)} \underbrace{A'(x)e^x + B'(x)e^{2x}}_{=\text{ 0 by assumption (1)}} + A(x)e^x + 2\cdot B(x)e^{2x} = A(x)e^x + 2\cdot Be^{2x}. \tag{*}$$

Next, we compute $\frac{d^2}{d^2x}y_p(x)$.

$$\begin{aligned}\frac{d^2}{d^2x}y_p(x) &= \frac{d}{dx}\left(\frac{d}{dx}y_p(x)\right)\\ &=^{(*)} \frac{d}{dx}\left(A(x)e^x + 2\cdot B(x)e^{2x}\right)\\ &= A'(x)e^x + 2\cdot B'(x)e^{2x} + A(x)e^x + 4\cdot B(x)e^{2x} \end{aligned} \tag{**}$$

In total, we obtain

$$\begin{cases} y(x) = A(x)e^x + B(x)e^{2x} \\ \frac{d}{dx}y(x) = A(x)e^x + 2\cdot Be^{2x} & \text{(by *)} \\ \frac{d^2}{d^2x}y(x) = A'(x)e^x + 2\cdot B'(x)e^{2x} + A(x)e^x + 4\cdot B(x)e^{2x}. & \text{(by **)} \end{cases}$$

Plugging this into the inhomogeneous differential equation gives

$$\begin{aligned} e^x &= \frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x)\\ &= \overbrace{A'(x)e^x + 2\cdot B'(x)e^{2x} + A(x)e^x + 4\cdot B(x)e^{2x}}^{=\frac{d^2}{d^2x}y(x)} - 3\overbrace{\left(A(x)e^x + 2\cdot B(x)e^{2x}\right)}^{=\frac{d}{dx}y(x)} + 2\overbrace{\left(A(x)e^x + B(x)e^{2x}\right)}^{=y(x)}\\ &= A'(x)e^x + 2\cdot B'(x)e^{2x} + A(x)e^x\underbrace{(1-3+2)}_{=0} + B(x)e^{2x}\underbrace{(4-3\cdot 2+2)}_{=0}\\ &= A'(x)e^x + 2\cdot B'(x)e^{2x}. \end{aligned} \tag{2}$$

Now that we have two equations (1), (2) in the two variables $A'(x), B'(x)$, we may express it as a system of linear equations,

$$\begin{cases} A'(x)e^x + B'(x)e^{2x} = 0 & (1) \\ A'(x)e^x + 2\cdot B'(x)e^{2x} = e^x & (2) \end{cases} \implies \begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix}\begin{pmatrix} A'(x) \\ B'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ e^x \end{pmatrix}.$$

which has the solution vector $\binom{A'(x)}{B'(x)} = \binom{-1}{e^{-x}}$. To find $A(x), B(x)$, we integrate disregarding the constants [4] to get

$$A'(x) = -1 \implies A(x) = -x$$
$$B'(x) = e^{-x} \implies B(x) = -e^{-x}.$$

That in turn yields $y_p(x) = \overbrace{-xe^x - e^x}^{:=A(x)e^x + B(x)e^{2x}}$. Utilising the homogeneous solution $y_h(x)$ from the previous problem, (1.5) allows us to write

$$y(x) = \overbrace{Ae^x + Be^{2x}}^{y_h(x)} + \overbrace{-xe^x - e^x}^{y_p(x)}$$
$$= \boxed{Ae^x + Be^{2x} - xe^x \quad \text{for } A, B \in \mathbb{R}.}$$

$\boxed{\xi}$

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c) *Consider the second-order inhomogeneous differential equation*

$$\frac{d^2}{d^2x}y(x) - 3\frac{d}{dx}y(x) + 2y(x) = e^x.$$

Find the specific solution that satisfies the conditions $y(0) = 1$ *and* $(\frac{d}{dx}y)(0) = 0$.

Solution. Simply, one combines the result from the previous exercise

$$y(x) = Ae^x + Be^{2x} - xe^x$$

with the initial conditions to obtain a system of equations

$$1 = y(0) = A + B$$
$$0 = y'(0) = Ae^0 + 2Be^{2\cdot 0} - e^0 - 0 \cdot e^0$$
$$= A + 2B - 1$$

which has the solutions $A = -3, B = 2$. We write

$$\boxed{y(x) = -3e^x + 2e^{2x} - xe^x.}$$

$\boxed{\xi}$

[4] We can do this, since the constants are accounted for in the homogeneous equation. For example, if $A(x) = -x + c$ then $A(x)e^x = -xe^x + ce^x$ which we combine with the constant term Ae^x from $y_h(x) = Ae^x + Be^{2x}$.

Exercise 2 *The logistic equation is a model for population growth with a maximum sustainable population. It is given by*

$$\frac{d}{dt}P(t) = rP(t)\left(1 - \frac{P(t)}{K}\right),$$

where $P(t)$ denotes the size of the population at time t, $r > 0$ is the (constant) growth rate of the population, and $K > 0$ is the maximum sustainable population.

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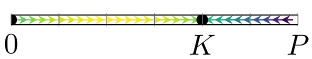
a) *Sketch the direction field for the logistic equation.*

Sketch. An ordinary differential equation $\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t))$ is defined by a vector field f. In this case, the equation only has one variable, and thus the vector field is one-dimensional.[5] Moreover, f is given exactly by the logistic equation

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \;\; P \mapsto f(P) = rP(1 - \frac{P}{K}).$$

A very rough sketch of the vector field is therefore

Vector Field for Logistic Equation $f(P) = rP\left(1 - \frac{P}{K}\right)$



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b) *Explain in words how $P(t)$ changes when $P(t) \ll K$, $P(t) = K$, and $P(t) > K$. How does $P(t)$ behave for large times?*

Explanation. There are three cases to consider.

$P(t) \ll K$ | First, let it be clear that $P(t) \ll K$ means that $P(t)$ is significantly smaller than K. If this is the case, then the fraction $\frac{P(t)}{k}$ is very small, and may be neglected. Then, $1 - \frac{P(t)}{K}$ is close to 1, and

$$\frac{d}{dt}P(t) \approx rP(t).$$

This is the equation for exponential growth with rate r. This means that $P(t)$ grows *almost* exponentially *towards* K, and we may write $P(t) \approx e^{rt}$.

$P(t) = K$ | In that case, $1 - \frac{P(t)}{K} = 0$, and

$$\frac{d}{dt}P(t) = rP(t)(\cancelto{0}{1 - \frac{P(t)}{K}}) = 0$$

[5] Seek page 8 of Lectures 4,5 for examples.

implies that $P(t)$ is a constant, fixed value. This is a natural consequence, since the constant K does not depend on t. The population is then in a state of equilibrium at $P(t) = K$.

$P(t) > K$ | Finally, $1 - \frac{P(t)}{K} < 0$ gives

$$\frac{d}{dt}P(t) = \underbrace{rP(t)(1 - \frac{P(t)}{K})}_{<0} < 0$$

meaning that the population is too big, and decays *towards* $P(t) = K$.

In summary, the relation between $P(t)$ and K characterizes $\frac{d}{dt}P(t)$. Precisely: as t gets larger, the population $P(t)$ tends to a state of equilibrium $P(t) = K$. $\boxed{\xi}$

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c) *Find the solution to the logistic equation with initial condition* $P(0) = P_0$.

Solution. First, let us re-write the equation as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

Next, separate the variables and rewrite

$$\frac{dP}{P \cdot (1 - \frac{P}{K})} = r\ dt \qquad \text{as} \qquad \frac{K}{P \cdot (K - P)} \cdot dP = r\ dt.$$

Partial Fractions. The left-hand side is difficult to integrate in this form. It would be much easier if we could write it as two terms $\frac{A}{P}$ and $\frac{B}{K-P}$ for some constants A, B. Luckily, the partial fraction method provides just that. Assume indeed that

$$\frac{K}{P \cdot (K - P)} = \frac{A}{P} + \frac{B}{K - P} \overset{\times P(K-P)}{\Longrightarrow} K = A \cdot (K - P) + B \cdot P \implies K = P \cdot (B - A) + A \cdot K.$$

Notice that the first statement in this chain of implications is an identity[6] on P, and thus we may plug-in $P = 0$ in the final statement to get

$$P \cdot (B - A) + A \cdot K = K \overset{P=0}{\Longrightarrow} A = 1.$$

By substituting $A = 1$ into the equation and solving for B, we get that

$$P \cdot (B - 1) + K = K \implies B = 1$$

As an exercise, you may check that $\frac{K}{P \cdot (K - P)} = \frac{1}{P} + \frac{1}{K - P}$.

[6] is true for all P.

Integration. With the partial fractions expression, we write

$$\frac{K}{P \cdot (K-P)} \cdot dP = r\ dt \quad \overset{\text{Partial Fractions}}{\Longrightarrow} \quad \frac{1}{P} + \frac{1}{K-P} = r\ dt \implies \int \frac{1}{P} dp + \int \frac{1}{K-P}\ dp = \int r\ dt$$
$$\implies \ln(P) - \ln(K-P) = rt + C \implies \ln\left(\frac{P}{K-P}\right) = rt + C.$$

$\mathbf{P(t) =?}$ Solving for P, we get

$$\ln\left(\frac{P}{K-P}\right) = rt + C \implies \frac{P}{K-P} = e^{rt} \cdot e^{C} \implies P(t) = \frac{Ke^{rt} \cdot e^{c}}{1 + e^{rt} \cdot e^{C}} \quad \text{for} \quad C \in \mathbb{R}.$$

Initial Condition. To find the constant C (rather e^C), we utilise the initial condition $P(0) = P_0$ to write

$$P_0 = P(0) = \frac{Ke^{r \cdot 0} \cdot e^{C}}{1 + e^{r \cdot 0} \cdot e^{C}} = \frac{Ke^{C}}{1 + e^{C}} \implies e^{C} = \frac{P_0}{K - P_0}.$$

Plug this into the expression to get

$$P(t) = \frac{Ke^{rt} \cdot A}{1 + e^{rt} \cdot A} \quad \text{for } A = \frac{P_0}{K - P_0}. \qquad \boxed{\xi}$$

Exercise 3 *Consider the system of ODEs*

$$\begin{cases} \frac{d}{dt}x(t) = x(t) - y(t) \\ \frac{d}{dt}y(t) = x(t) + y(t). \end{cases}$$

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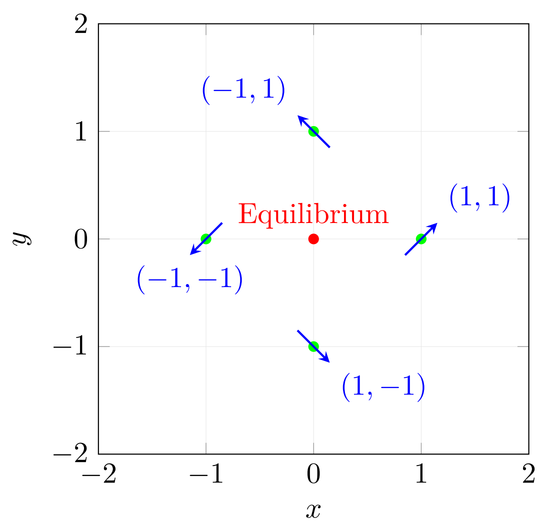
a) *Sketch the vector field.*

Sketch. Let us start with some observations, and compute gradient at different vectors $\begin{bmatrix} x \\ y \end{bmatrix}$.

1. $(x, y) = (1, 0) \implies \begin{cases} \frac{dx}{dt} = 1 - 0 = 1 \\ \frac{dy}{dt} = 1 + 0 = 1 \end{cases}$
2. $(x, y) = (0, 1) \implies \begin{cases} \frac{dx}{dt} = 0 - 1 = -1 \\ \frac{dy}{dt} = 0 + 1 = 1 \end{cases}$
3. $(x, y) = (-1, 0) \implies \begin{cases} \frac{dx}{dt} = -1 - 0 = -1 \\ \frac{dy}{dt} = -1 + 0 = -1 \end{cases}$
4. $(x, y) = (0, -1) \implies \begin{cases} \frac{dx}{dt} = 0 - -1 = 1 \\ \frac{dy}{dt} = 0 + -1 = -1 \end{cases}$

A positive gradient in x indicates growth in the $x-$direction, and a negative gradient in y indicates decay in the $y-$direction. This is a good point to stop and observe some nice drawings.

Vector Field Directions − I

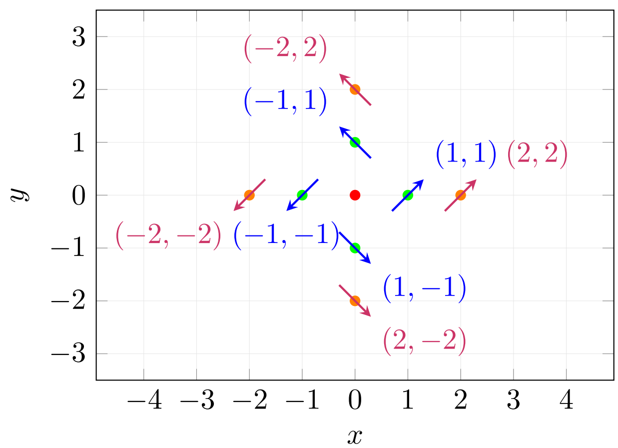


It seems that there is a tendency to go counter-clockwise. It is not clear just yet whether the trajectories converges inward or diverges outward. For this, let us compute the gradient for vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with greater magnitude.

1. $(x, y) = (2, 0) \implies \begin{cases} \frac{dx}{dt} = 2 - 0 = 2 \\ \frac{dy}{dt} = 2 + 0 = 2 \end{cases}$
2. $(x, y) = (0, 2) \implies \begin{cases} \frac{dx}{dt} = 0 - 2 = -2 \\ \frac{dy}{dt} = 0 + 2 = 2 \end{cases}$
3. $(x, y) = (-2, 0) \implies \begin{cases} \frac{dx}{dt} = -2 - 0 = -2 \\ \frac{dy}{dt} = -2 + 0 = -2 \end{cases}$
4. $(x, y) = (0, -2) \implies \begin{cases} \frac{dx}{dt} = 0 - -2 = 2 \\ \frac{dy}{dt} = 0 + -2 = -2 \end{cases}$

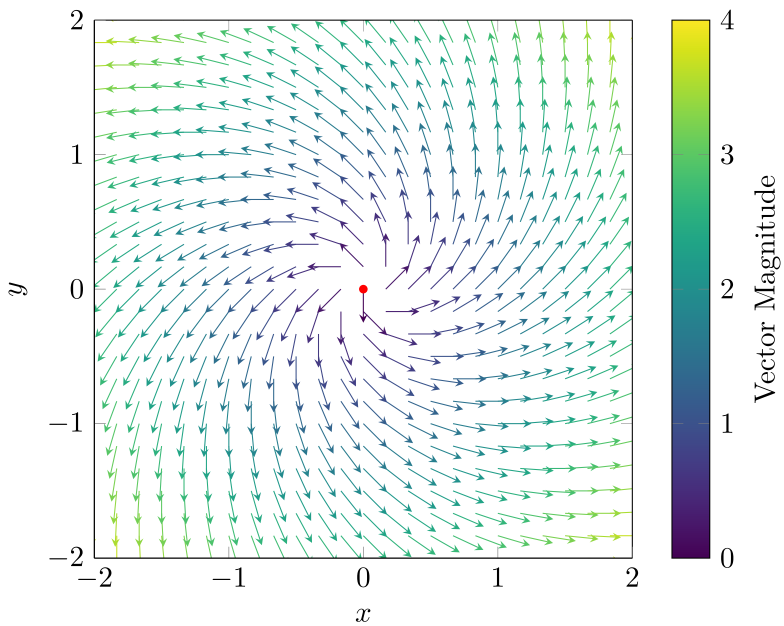
The gradients seem to get greater in magnitude. This indicates an *unstable* vector field whose trajectories diverge outwards with time.

Vector Field Directions − II



With these observations, the vector field should take on the form

Vector Field for $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$



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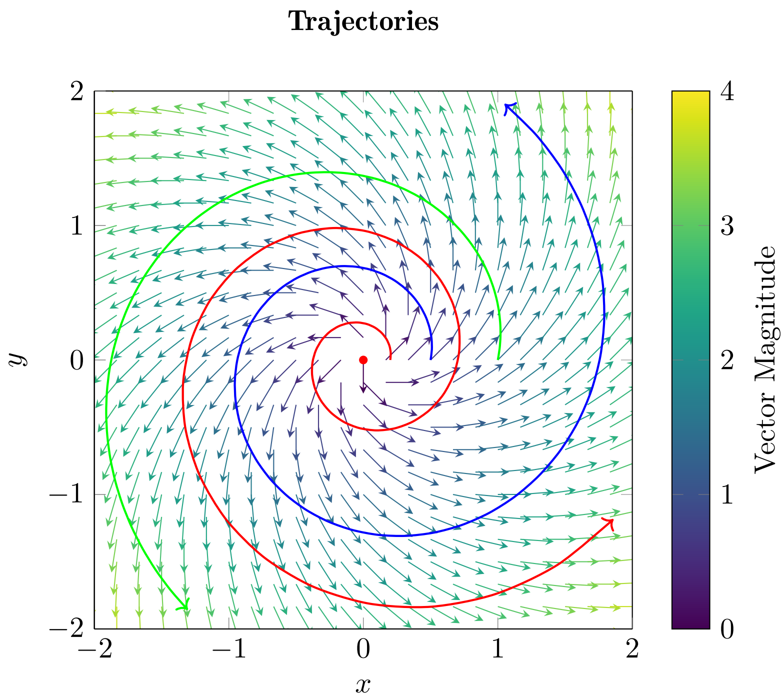
b) *Consider the system of ODEs*

$$\begin{cases} \frac{d}{dt}x(t) = x(t) - y(t) \\ \frac{d}{dt}y(t) = x(t) + y(t). \end{cases}$$

Using the vector field sketch, sketch a few representative solution trajectories in the phase space.

Sketch. Using our vector field sketch, we pick some starting points and see where the vector field flows them to.



$\boxed{\xi}$

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c) *Consider the system of ODEs*

$$\begin{cases} \frac{d}{dt}x(t) = x(t) - y(t) \\ \frac{d}{dt}y(t) = x(t) + y(t). \end{cases}$$

Determine if the orbits are periodic.
Hint: Look at the arrows in the vector field as you move away from the origin. Do they drive you further away, or do they guide you back toward a loop? If the arrows push you away, the orbits are not periodic.

Solution. Clearly, and as demonstrated above, the orbits are not periodic. $\boxed{\xi}$

Exercise 3, but Analytically – Bonus *Consider the system of ODEs*

$$\begin{cases} \frac{d}{dt}x(t) = x(t) - y(t) \\ \frac{d}{dt}y(t) = x(t) + y(t). \end{cases}$$

Describe the analytical behaviour of this system.

Description. First, observe that the system of equations is linear, which is very nice. Let us combine the two first-order differential equations into one degree-two differential equation.

$$x' = x - y \overset{\frac{d}{dt}}{\Longrightarrow} x'' = \overbrace{x'}^{:=x-y} - \underbrace{y'}_{:=x+y} = -2 \cdot \underbrace{y}_{:=x-x'} = -2(x - x') \implies x'' + 2x' - 2x = 0$$

This is exactly the same setup as in **1a)**. Proceed with the choice of $x = e^{\lambda t}$, and let us solve for λ.

$$x'' + 2x' - 2x = 0 \overset{x(t)=e^{\lambda t}}{\Longrightarrow} e^t(\lambda^2 + 2\lambda - 2) = 0 \implies \lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} = 1 \pm i.$$

Next, the following remark is quite useful,

Remark. If $\lambda_{1,2} = \alpha + i\beta$ are two complex solutions to the characteristic equation, then

$$e^{\alpha t}\cos(\beta t) \qquad \& \qquad e^{\alpha t}\sin(\beta t)$$

are two linearly independent solutions.[7]

The general solution $x(t)$ is the linear combination of all independent solutions. The remark thus allows us to write the solutions for $\alpha = 1, \beta = 1$ to get

$$x(t) = e^t(A\cos t + B\sin t).$$

Next, $y = x - x'$ gives

$$y(t) = \cancel{e^t(A\cos t + B\sin t)} - \Big(\cancel{e^t(A\cos t + B\sin t)} + e^t(-A\sin t + B\cos t)\Big) = e^t(A\sin t - B\cos t).$$

With the solution

$$\begin{cases} x(t) = e^t(A\cos t + B\sin t) \\ y(t) = e^t(A\cos t - B\sin t) \end{cases}$$

in hand, let us attempt to study the behaviour of $x(t), y(t)$ with time. First, the solution is unstable in the sense that its magnitude grows exponentially with time; credited to the factor e^t. The $\cos t, \sin t$ factors add a counter-clockwise rotation to the field. To conclude, the trajectories spiral outwards in a counter−clockwise direction. The vector $\vec{0}$ must therefore be the only equilibrium point. $\boxed{\xi}$

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[7] We know this from class. See the Remark on Page 14, Lectures 2 and 3 on moodle.

Sheet Three

Keywords

Phase Space Trajectory, Vector Field, Conservation Law of Energy, Harmonic Oscillator, Global and Local Lipschitz Continuity.

Contents

Exercise 1 *Let the velocity be defined as $v(t) = \frac{d}{dt}x(t)$.*

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a) *Express the second-order differential equation $\frac{d^2}{d^2t}x(t) + \frac{k}{m}x(t) = 0$, which models an undamped harmonic oscillator without external forcing, as an equivalent system of first-order ordinary differential equations using the velocity variable $v(t)$.*

Solution. If $v = x'$ as the question assumes, then $v' = x''$ and we may re-write the second-order differential equation as a system of two equations, both of first-order. $\boxed{\xi}$

$$x' = v \tag{iii.1}$$

$$v' = -\frac{k}{m} \cdot x \tag{iii.2}$$

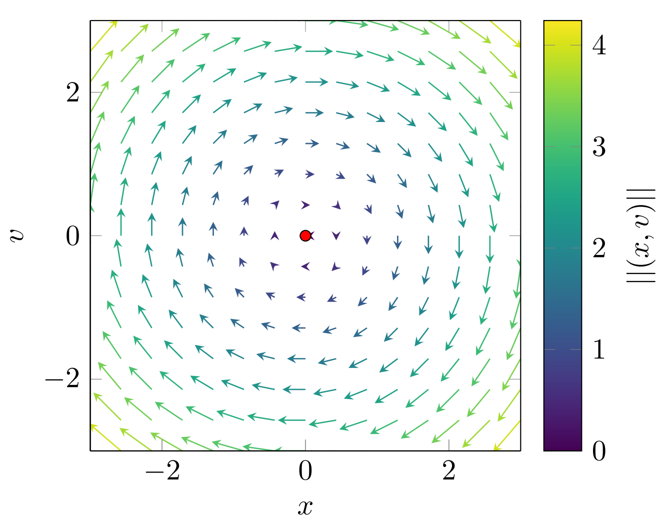
---

b) *Sketch the vector field and phase portrait corresponding to the system of first-order ODEs from part a) for the parameter values $k = 2$ and $m = 2$.*

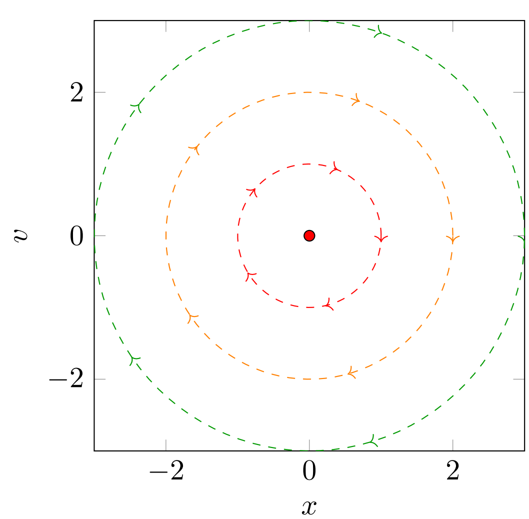
Remark. The choice $k = m = 2$ gives rise to $\begin{cases} x' = v \\ v' = -x \end{cases}$. This is precisely **Example 1** from the lecture, differing only by a minus sign.[1] This difference is, in fact, reflected in the direction of rotation, which is clockwise compared to anti-clockwise sketch of the example. There, a similar phase portrait is offered as well. We proceed nevertheless without this knowledge.

Sketch. The system corresponding to the choices $k = m = 2$ is given by $f(x, v) = \binom{v}{-x}$. To plot its associated vector field and phase portrait, one simply computes the gradient at a few points in the $xv-$plane to get the vector field, then traces some trajectories along these gradient vectors. [2]

Vector Field Sketch



Phase Portrait



$\boxed{\xi}$

[1]See Page 8, Section 1.3, Lectures 4 & 5

[2]See (**ii.3**) for more on this technique.

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c) *Sketch the vector field and phase portrait for the same system with parameter values $k = 8$ and $m = 2$. Describe how and why the phase portrait changes when the parameter k is increased.*

Sketch. The system of equations $\begin{cases} x' &= v \\ v' &= -\dfrac{k}{m} \cdot x \end{cases}$ tells us that $\dfrac{dx}{dt} = v$ and $\dfrac{dv}{dt} = -\dfrac{k}{m} \cdot x$. Since both the vector field and phase portrait live in the $xv-$plane, it is a good idea to eliminate the time component. For this, compute $\frac{dv}{dx}$ using the chain rule to get $\frac{dv}{dt} \cdot \frac{dt}{dx} = \frac{-k}{m} \cdot \frac{x}{v}$. This is a separable differential equation!

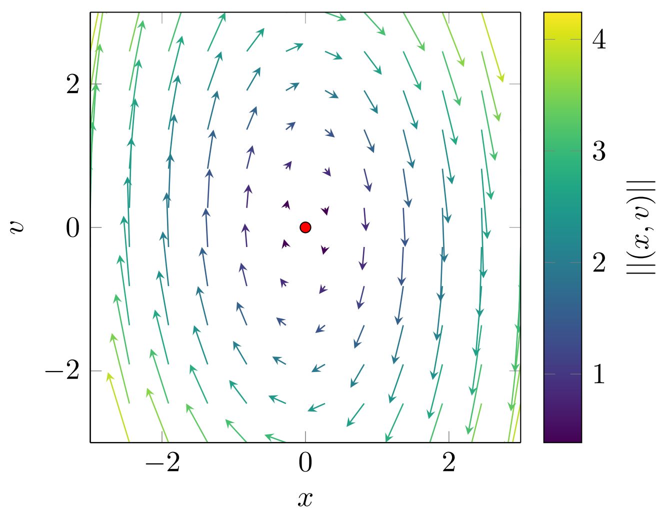
$$v\ dv = -\frac{k}{m} \cdot x\ dx \overset{\int \cdots}{\Longrightarrow} \frac{v^2}{2} = \frac{-k}{m} \cdot \frac{x^2}{2} + C \overset{\times 2}{\Longrightarrow} v^2 = -\frac{k}{m} \cdot x^2 + 2C \overset{+\frac{k}{m} \cdot x^2}{\Longrightarrow} v^2 + \frac{k}{m} \cdot x^2 = 2C$$
$$\overset{\times \frac{m}{2}}{\Longrightarrow} \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = mC$$

For physicists, this should be familiar! The expression $\frac{1}{2} \cdot mv^2$ encodes potential energy, whereas kinetic energy is displayed as $\frac{1}{2} \cdot kx^2$. This is the conservation law of energy. To see this, set $E = mC$ and write

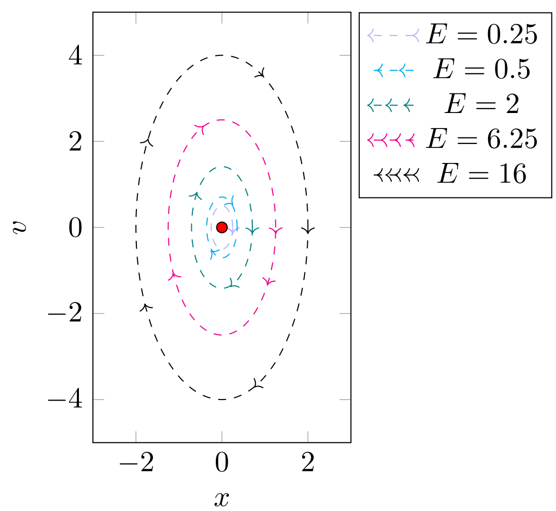
$$\begin{aligned} \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t) &= E(t) \\ &= E(0) := \frac{1}{2}mv^2(0) + \frac{1}{2}kx^2(0). \end{aligned}$$

The equation $E(t) = E(0)$ encodes that the initial total energy $E(0)$ is preserved as time flows. All points of the solutions $\binom{x}{v}$ with initial condition $\binom{x_0}{v_0}$ should therefore lie on an ellipse.[3]

Vector Field Sketch



Phase Portrait



To study the change in phase portrait upon varying k, consider the horizontal endpoints of the ellipse on the $x-$axis. There, we have $v = 0$ and $E = \frac{1}{2}kx^2$. The energy is constant[4], therefore increasing k implies that x^2 must decrease. This forces both endpoints to get closer to the origin. You may convince yourself by a similar argument that increasing m shifts the ellipse in the vertical direction. $\boxed{\xi}$

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[3] The conservation law of energy is an equation of an ellipse. Do you see this?
[4] by the conservation law

Exercise 2 *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{\frac{2}{3}}$.*

Before commencing with the proof, let us give two precise definitions.[5]

Definition 1.13 (Global Lipschitz Continuity)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *locally* Lipschitz continuous if there exists an $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}$.

Definition 1.13 (Local Lipschitz Continuity)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *globally* Lipschitz continuous if for every x_0 we may find a neighbourhood $\mathcal{U}_{x_0}$ around it such that

$$|f(x) - f(y)| \leq L_{x_0}|x - y|$$

for all $x, y \in \mathcal{U}_{x_0}$. The subscript $_{x_0}$ signifies the dependence of L on x_0.

Next, proceed to prove the following statements.

---

a) *Show that f is not locally Lipschitz continuous.*

Proof. First, notice that $f(x) := x^{\frac{2}{3}} = \sqrt[3]{x^2}$ behaves not-so-nicely near $x = 0$. Formally, the derivative

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

exists for $x \neq 0$, and is unbounded[6] as x approaches 0. This makes it a possible candidate point to exploit. Proceed, and suppose for the sake of contradiction that f is Lipschitz. Then this suggests that we may find $\delta,\ L_{x_0} > 0$ such that

$$|x^{\frac{2}{3}} - y^{\frac{2}{3}}| \leq L_{x_0}|x - y|.$$

for all $x, y \in (-\delta, \delta)$. To utilise our earlier observation, set $y = 0$ and let $x \to 0^+$ to get

$$x^{\frac{2}{3}} \leq L_{x_0}x \overset{\times x^{-1}}{\Longrightarrow} x^{-\frac{1}{3}} \leq L_{x_0} \tag{*}$$

for some constant L. Notice, however, that (*) implies that

$$\infty = \lim_{x \to 0^+} x^{-\frac{1}{3}} \leq L_{x_0}. \quad \boxed{\lightning}$$

Clearly, there is no constant L_{x_0} that works, thus the assumption fails. $\boxed{\xi}$

[5] Do you remember this from Analysis I? If not, seek Page 1, Lecture 8

[6] $|f'(x)| \to \infty$

---

b) *Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$. Show that g is locally Lipschitz continuous but not globally Lipschitz continuous.*

Proof. To show that g is locally Lipschitz continuous, choose $x_0 \in \mathbb{R}$, and let $\mathcal{U}_{x_0} = [x_0 - \delta, x_0 + \delta]$ for some positive δ. Notice that Lipschitz continuity is equivalent to

$$|g(x) - g(y)| \leq L_{x_0}|x - y| \iff \frac{|g(x) - g(y)|}{|x - y|} \leq L_{x_0}.$$

Proceed with yet another observation $-$ g is continuous on the closed interval $[x_0 - \delta, x_0 + \delta]$ and differentiable on the open interval $(x_0 - \delta, x_0 + \delta)$. Apply the mean value theorem to establish the existence of some $\xi \in [x_0 - \delta, x_0 + \delta]$ for which

$$g'(\xi) = \frac{|g(x) - g(y)|}{|x - y|}$$

for all $x, y \in [x_0 - \delta, x_0 + \delta]$. Combining both facts, we notice that

$$\frac{|g(x) - g(y)|}{|x - y|} = g'(\xi) \leq L_{x_0}.$$

To bound the derivative $g'(x) = 3x^2$ over $[x_0 - \delta, x_0 + \delta]$, we note that $\max(|x_0 - \delta|, |x_0 + \delta|)$ maximizes g. Therefore, the choice of

$$L_{x_0} := \max(|x_0 - \delta|, |x_0 + \delta|)$$

gives the result. Observe how L always depends on the choice of x_0. In that respect, it is not universal.

In a style similar to **a)**, we show that g is not globally Lipschitz. Assume for the sake of contradiction that g is globally Lipschitz, then we establish the existence of L for which

$$|x^3 - y^3| \leq L|x - y|.$$

for any $x, y \in \mathbb{R}$. Therefore, it would not cause an issue if one makes the choice of $y = 0$ to get that

$$|x^3| \leq L \cdot |x| \overset{\times|x^{-1}|}{\Longrightarrow} |x^2| \leq L.$$

The implication is clearly false, since the statement should hold for all $x \in \mathbb{R}$. Taking $|x| \longrightarrow \infty$ gives the contradiction. ⌇

Exercise 3 *Prove that every continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous. Hint: One possibility is to use the inequality $\left| \int_x^y g(t)\, dt \right| \leq \int_x^y |g(t)|\, dt$.*

The extreme value theorem in higher dimensions is stated later in Lectures 14 & 15.

Theorem 2.24 (Extreme Value Theorem)**.** *If $f : S \to \mathbb{R}^d$ is continuous on a closed and bounded set $S \subseteq \mathbb{R}^d$, then f attains a minimum and maximum value on S. Precisely, one writes*

$$(\forall \vec{x} \in S)(\exists\ \vec{a}, \vec{b} \in S)\ :\ f(\vec{a}) \leq f(\vec{x}) \leq f(\vec{b}).$$

It is a good idea nevertheless to include it here, as it recaps the one-dimensional case.

Proof. Given x_0, the goal is to show that we can find a neighbourhood $\mathcal{U}_{x_0}$ on which the Lipschitz condition is satisfied. Assume indeed that f is continuously differentiable. Then, f' is continuous, and for $x_0 \in \mathbb{R}$ we may choose $\mathcal{U}_{x_0} = [x_0 - \delta, x_0 + \delta]$ for some $\delta > 0$. Theorem 2.24 gives an upper bound

$$|f'(t)| \leq M_{x_0} \tag{*}$$

for every $t \in \mathcal{U}_{x_0}$. Notice that for fixed δ, the value of M_{x_0} depends on x_0. Next, write

$$f(y) - f(x) = \int_x^y f'(t)\ dt \qquad \text{(fundamental theorem of calculus)}$$

for $x, y \in \mathcal{U}_{x_0}$. We are yet to utilize the hint. To account for this shortcoming, write

$$|f(y) - f(x)| = \left| \int_x^y f'(t)\ dt \right| \leq \int_x^y |f'(t)|\ dt \overset{(*)}{\leq} M_{x_0} \cdot (y - x) \leq M_{x_0} \cdot |x - y|.$$

This statement is true for all $x, y \in \mathcal{U}_{x_0}$ with $x \leq y$. The point x_0 was arbitrarily chosen, therefore f must be locally Lipschitz. $\boxed{\xi}$
