Constructor University **Spring semester 2023**

Calculus and Elements of Linear Algebra II

Final exam

Last Name

First Name

Matriculation Number

Please note that the following instructions apply for the final exam:

- Solve all your exercises on paper, and
  - **Remote students:** upload all your notes to LPlus at the end.
  - **On-campus students:** submit all your notes.
- The grade of the exam will solely be based on your returned notes.
- Use a separate sheet of paper for each exercise.
- Your notes must be clearly legible.
- Show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- There are 6 exercises with 15 points each in this exam. 10 points count as bonus points, i.e., your grade will be computed as a percentage based on a total of 80 points.

Problem 1: One-variable Calculus [15 points]

(a) Verify if the following series converge: $\sum_{k=1}^{\infty} \frac{k^2-1}{4^k}$

(b) Determine the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

(c) Write the Taylor series for the function $f(x) = x\cos 3x$

(d) Find first three non-zero terms in the Taylor series of the function $f(x) = e^{-x^2}\cos(x)$

Problem 2: Multi-variable Calculus [15 points]

(a) Consider the function $f(x, y) = e^{x^2y-x}$. Compute its gradient and derivatives $\partial_x\partial_y f$ and $\partial_y\partial_x f$. Compute at point $\vec{a} = (2, 0)$ the directional derivative of f in the direction $\frac{1}{\sqrt{2}}(-1, 1)$

(b) Let $f(x, y) = e^{x^2} \cos y$. Find the second order Taylor expansion of f around the point $(0, 0)$.

(c) Compute the differential df of the function $f(x, y, z) = \ln(x + 2y + 3z)$.

Problem 3: Critical Points [15 points]

(a) What are the local minima, local maxima, and saddle points of $f(x,y) = 8 + 2x^3 + 2y^3 - 6xy$?

(b) Let $f(x,y) = 1 + xy$. Using the method of Lagrange multipliers, find the critical points of f on the unit circle, i.e., under the constraint $G(x,y) = x^2 + y^2 - 1 = 0$.

Problem 4: Ordinary Differential Equations [15 points]

(a) Solve the differential equation $y' = -y/x$, $x \neq 0$.

(b) Solve the differential equation $(1 + x^2)dy - 2xydx = 0$. Find the partial solution satisfying the initial condition $y(0) = 1$.

Problem 5: Determinant, Eigenvalues, and Eigenvectors [15 points]

(a) Find the determinant and the trace of the following matrix:

$$A = \begin{pmatrix} 3 & 3 & -1 \\ 4 & 1 & 3 \\ 1 & -2 & -2 \end{pmatrix}$$

(b) Consider the following system of linear equations:

$$\begin{cases} 3x_1 + x_2 + 2x_3 & = & 4 \\ -x_1 + 2x_2 - 3x_3 & = & 1 \\ -2x_1 + x_2 + x_3 & = & -2 \end{cases}$$

Use Cramer's rule to determine the solution x_1.

(c) Find the eigenvalues and the eigenvectors of the following matrix:

$$B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

Problem 6: Linear differential equations of higher orders [15 points]
Consider the following equation:

$$y''' - 4y' = 24e^{2t}$$

(a) Write the corresponding homogeneous equation and solve it.

(b) Find the full solution of the given inhomogeneous equation.

(c) Find the partial solution that satisfies initial conditions $y(0) = 1, y'(0) = 1, y''(0) = 0$.