Final Exam 2024: 18th of December, 12:30-14:30

Idea: Practice!

< 2020 Midterm, Prof. Marcel Oliver >

Calculus and Elements of Linear Algebra I

Mock Midterm Exam

Monday, November 2, 2020

1. Compute the following limits, if they exist. Else, argue why the limit does not exist.

   (a) $\displaystyle\lim_{s\to-1} \frac{\frac{1}{s}-1}{s^3-1}$

   (b) $\displaystyle\lim_{x\to\infty} \frac{\mathrm{e}^{2x}+x^3+\ln x}{3\,\mathrm{e}^{2x}-x^3+\cos x}$

   (c) $\displaystyle\lim_{r\to1} \frac{|r-1|}{r^2-1}$

   (5+5+5)

2. The function $f(x)$ is defined on the interval $[0,2]$ and is between $4-x$ and x^2+2 for all x in this interval. Does it have to be continuous at $x=1$? Explain why or why not. (5)

3. Show that the equation $x^7-3\,x-1=0$ has at least one solution in the interval $[-1,1]$. (5)

4. (a) Show that
$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}\,.$$
   (b) Consider the function
$$f(x) = 2\,\arctan x - x\,.$$
   Find its domain, horizontal and vertical asymptotes, local minima, local maxima, and inflection points of f. Identify the regions where the graph of f is concave upward or concave downward. Finally, sketch the graph of the function.

   (5+10)

5. An airplane is flying towards a radar station at a constant height of 6 km above the ground. The distance s between the airplane and the radar station is decreasing at a rate of 400 km/h when $s=10$ km. What is the horizontal speed of the plane? (10)

6. Compute the following definite or indefinite integrals.

   (a) $\displaystyle\int x^{-3}\,\mathrm{e}^{1/x}\,\mathrm{d}x$

   (b) $\displaystyle\int \frac{x+1}{x^2\,(x^2+1)}\,\mathrm{d}x$

   (c) $\displaystyle\int_0^{2\pi} (\cos^2\phi - \sin^2\phi)\,\mathrm{d}\phi$

(10+10+5)

7. Find the derivative of the function

$$F(x) = \int_{\sqrt{x}}^{x} \frac{\mathrm{e}^t}{t}\,\mathrm{d}t\,.$$

(5)

#1

1a) $\lim\limits_{s\to -1} \dfrac{\frac{1}{s}-1}{s^3-1} = \dfrac{-1-1}{-1-1} = 1$

1b) $\lim\limits_{x\to\infty} \dfrac{e^{2x}+x^3+\ln x}{3e^{2x}-x^3+\underbrace{\cos x}_{[-1,1]}} = \lim\limits_{x\to\infty} \dfrac{\cancel{e^{2x}}}{3\cancel{e^{2x}}} = \dfrac{1}{3}.$

e^x grows faster than any Poly.

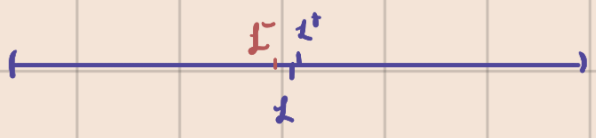
$\ln x$ grows slower than any Poly.

$$\frac{\overset{1}{\dfrac{e^{2x}}{e^{2x}}} + \overset{0}{\dfrac{x^3}{e^{2x}}} + \overset{0}{\dfrac{\ln(x)}{e^{2x}}}}{\overset{3}{\dfrac{3\cancel{e^{2x}}}{\cancel{e^{2x}}}} - \overset{0}{\dfrac{x^3}{e^{2x}}} + \overset{0}{\dfrac{\cos x}{e^{2x}}}}$$

1c) $\lim\limits_{r\to 1}\left(\dfrac{|r-1|}{r^2-1} = \dfrac{\pm\cancel{(r-1)}}{\cancel{(r-1)}(r+1)} = \dfrac{\pm 1}{(r+1)}\right) = \dfrac{\pm 1}{2} = \pm\dfrac{1}{2}$

$\lim\limits_{r\to 1}\left(\dfrac{|r-1|}{r^2-1} = \dfrac{|r-1|}{(r-1)(r+1)}\right)$

$\lim\limits_{r\to 1^+} \dfrac{\cancel{|r-1|}}{\cancel{(r-1)}(r+1)} = \dfrac{1}{2}$ and $\lim\limits_{r\to 1^-} \dfrac{+\cancel{(r-1)}}{-\cancel{(r-1)}(r+1)} = -\dfrac{1}{2}$



$|x-1| \leftarrow \begin{cases} x-1 \\ -(x-1) \end{cases}$

$\Rightarrow$ limit does not exist.

$\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = \dfrac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = 2$

or

$\lim\limits_{x\to 1} \dfrac{x^2-1}{-(x-1)} = \dfrac{\cancel{(x-1)}(x+1)}{-\cancel{(x-1)}} = \dfrac{1+1}{-1} = -2$

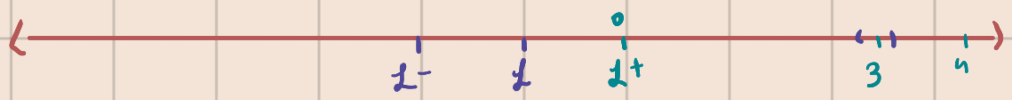
#2 $f : [0,2] \to \mathbb{R}$

$f(x) \in [4-x, x^2+2]$?

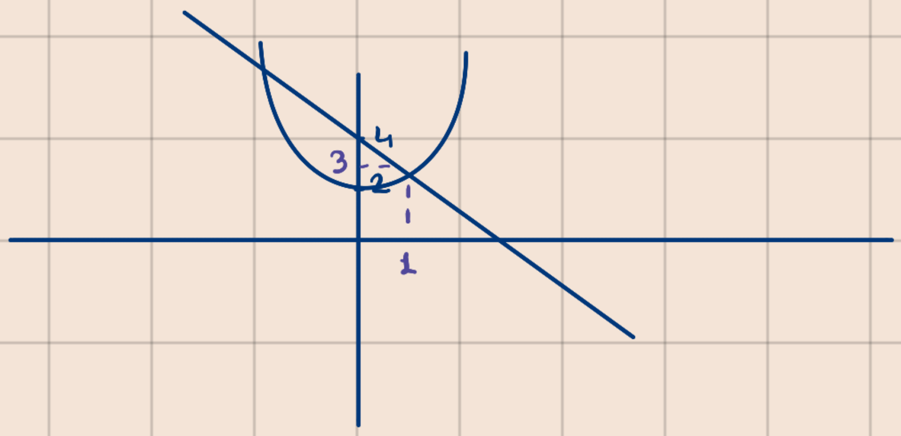
f cont. at $x=1 \Rightarrow$ Upper & lower limit match ! Inspecting...

$f(1) \in [4-1, 1^2+2] = [3,3] \Rightarrow \boxed{f(1) = 3}$ MUST ?

$\lim\limits_{x \to 1^+} f(x) \in [4-1^+, (1^+)^2+2] = [3^-, 3^+] = [3,3]$



$\lim\limits_{x \to 1^-} f(x) \in [4-1^-, (1^-)^2+2] = [3^-, 3^+] = [3,3]$



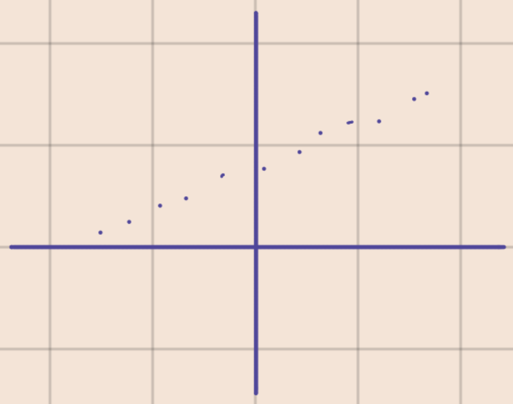
Exploring...

$f(0.2) \in [4-0.2, 0.2^2+2]$

$[2.004, 3.8]$

$f(2) \in [2,6]$, $f(0) = [2,4]$

$f(1) \in [3,3] = 3$



$x \to 1^+ \quad 3^- = 4-x \leq f(x) \leq x^2+1 = 3^+$

$x \to 1^- \quad 3^- = x^2+1 \leq f(x) \leq 4-x = 3^+$

#3 $f(x) = x^7 - 3x - 1 = 0$

$\boxed{\text{Idea:}}$

- draw f without lifting the pen (Continuity)
- go from the $-$ve side $\exists c_1 : f(c) < 0$
- to the $+$ve side $\exists c_2 : f(c) > 0$

- $f(x)$ is a polynomial (Continuous)
- $f(-1) = -1 + 3 - 1 > 0$
- $f(1) = 1 - 3 - 1 < 0$

$\Big\}$ check points in the interval!

Then the intermediate value theorem guarantees one crossing of the x-axis, i.e. one root.

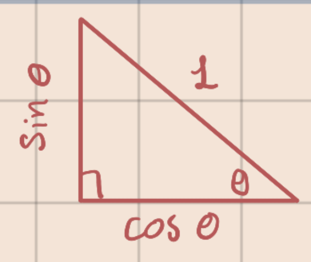
#4

4a) $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$,

$y = \arctan x = \tan^{-1} x$, $\boxed{\text{goal: find } \frac{dy}{dx}}$

$\tan y = \cancel{\tan}(\cancel{\tan^{-1}} x) = x \Rightarrow \tan y = x$

$$\frac{d}{dx}\left(\tan x = \frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \overbrace{(\sin x)'}^{\cos x} - \sin x \cdot \overbrace{(\cos x)'}^{-\sin x}}{\cos^2 x}$$



$\Downarrow$

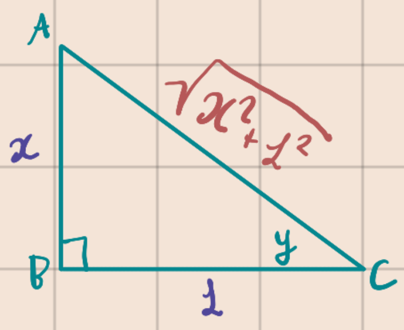
$\cos^2\theta + \sin^2\theta = 1$

$$= \frac{\cos x \cdot \cos x + \sin x \sin x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$\sec^2 y \cdot \boxed{\frac{dy}{dx}} = \frac{d}{dx} \tan y = \frac{d}{dx} x = 1 \cdot \frac{\cancel{d}x}{\cancel{d}x}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\frac{1}{\cos^2 y}} = \cos^2 y = \frac{1}{x^2+1}}$$



$\sqrt{x^2+1} = AC$

$\tan y = \frac{\text{opp}}{\text{adj}} = \frac{AB}{BC}$

$\sin y = \frac{\text{opp}}{\text{hyp}} = \frac{AB}{BC}$

$\cos y = \frac{\text{opp}}{\text{hyp}} = \frac{AB}{BC}$

(The trigonometric construction ....)

$y = \tan^{-1} x \Rightarrow \tan y = x$

$\cos^2 y = \left(\frac{1}{\sqrt{x^2+1}}\right)^2 = \frac{1}{x^2+1}$

Exercise. $\frac{d}{dx} \operatorname{arcsec} x$

#6

6a) $\int x^{-3} e^{\frac{1}{x}} \, dx$, inner function : $\frac{1}{x}$

$\int \frac{1}{x^3} e^{\frac{1}{x}} \, dx$, $\frac{d}{dx}$ inner function: $\frac{-1}{x^2}$ → probably should work.

$u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx$

$\Rightarrow dx = -x^2 du$

$$I = \int \frac{1}{x^{\cancel{3}}} \cdot e^u \; -\cancel{x^2} \, du$$

$$= \int \underbrace{\frac{1}{x}}_{=u!} e^u (-1) \, du = -\int u e^u \, du$$, different types parts

$u = u$, $dv = e^u du$ ⇒ integrate e^u, $\frac{d}{dx} u$

$\boxed{\int u \, dv = uv - \int v \, du}$ $I = \underbrace{u}_{u} \underbrace{e^u}_{v} - \int \underbrace{e^u}_{v} \underbrace{1 \, du}_{du} \boxed{= ue^u - e^u + C}$

6c) $\int_0^{2\pi} \cos^2\phi - \sin^2\phi \, d\phi$, $\cos^2\phi - \sin^2\phi = \cos 2\phi$

$$\int_0^{2\pi} \cos 2\phi \, d\phi = 0$$