Agenda.

0. More Derivatives
1. Examples
2. Implicit Differentiation
3. Application of 2^{nd} Derivative

0. More Derivatives.

[Without proof. :-]

$\rightarrow \frac{d}{dx} \ln(x) = \frac{1}{x}$

$\rightarrow \frac{d}{dx} x^n = n \cdot x^{n-1}$

$\rightarrow \frac{d}{dx} e^x = e^x$

$\rightarrow \frac{d}{dx} \tan(x) = \underbrace{\sec^2(x)}_{\text{"secant"}} := \frac{1}{\cos^2(x)}$

Recall:

$\rightarrow \frac{d}{dx} \sin(x) = \cos(x)$

$\rightarrow \frac{d}{dx} \cos(x) = -\sin(x)$

Results.

$\rightarrow \frac{d}{dx} \frac{f}{g} = \frac{gf' - fg'}{g^2}$ (quotient)

$\rightarrow \frac{d}{dx} \overbrace{f \circ g}^{= f(g(x))} = \overbrace{(f' \circ g)}^{= f'(g(x))} \cdot \overbrace{g'}^{g'(x)}$ (chain)

$\rightarrow \frac{d}{dx} fg = fg' + gf'$ (product)

$\rightarrow \frac{d}{dx} f^{-1} = \frac{1}{\underbrace{f' \circ f^{-1}}_{f'(f^{-1}(x))}}$ (inverse)

1. Examples.

a) $h(x) = \frac{3x^2+1}{x^5+x} = (3x^2+1)\cdot(x^5+x)^{-1}$

$\rightarrow \frac{d}{dx} fg = fg' + gf'$ (product)

$f = (3x^2+1) \Rightarrow f' = 6x$ $:= f' \circ g$

$g = \underbrace{(x^5+x)}^{-1} \Rightarrow g' = -(x^5+x)^{-2} \cdot (5x^4+1)$

$$\boxed{\begin{array}{l} (f' \circ g) \cdot g' \quad , \; x^{-1} \circ (x^5+x) \\ f = x^{-1} \Rightarrow f' = -x^{-2} \; ; \; g = x^5+x \Rightarrow g = 5x^4+1 \end{array}}$$

$h' = (3x^2+1) \cdot g' + (x^5+x)^{-1} \cdot 6x$

$\rightarrow \frac{d}{dx} \frac{f}{g} = \frac{gf' - fg'}{g^2}$ (quotient)

$f = 3x^2+1 \Rightarrow f' = 6x$

$g = x^5+x \Rightarrow g' = 5x^4+1$

$h' = \frac{(x^5+x)\,6x - (3x^2+1)(5x^4+1)}{(x^5+x)^2}$

b) $h(x) = \sin(\cos(x))$ $\begin{cases} f = \sin(x) \\ g = \cos(x) \end{cases}$

$$\boxed{\rightarrow \frac{d}{dx} f \circ g = (f' \circ g) \cdot g' \quad \text{(chain)}}$$

$h'(x) = \cos(\cos(x)) \cdot [\cos(x)]'$

$= -\cos(\cos(x)) \cdot \sin(x)$

$* [\sin(x)]' = \cos(x) \cdot x'$

$= \cos(x) \cdot 1$

$\Rightarrow$ We're always using Chain rule!

c) $h(x) = \sin(\cos(x^2+e^x))$ $\qquad g = \cos(x^2+e^x)$

$$[\cos(x^2+e^x)]' = -\sin(x^2+e^x)\cdot(2x+e^x) = g'$$

$\rightarrow \frac{d}{dx} f\circ g = (f'\circ g)\cdot g'$ (chain)

$$h'(x) = -\cos(\cos(x^2+e^x))$$
$$\cdot \sin(x^2+e^x)\cdot(2x+e^x)$$

d) $h(x) = e^{\cos(x)}\cdot \sin(x^2\cdot \ln(x))$

$\rightarrow \frac{d}{dx} fg = fg' + gf'$ (product)

$\rightarrow \frac{d}{dx} f\circ g = (f'\circ g)\cdot g'$ (chain)

$$[e^{\cos(x)}]' = e^{\cos(x)}\cdot[\cos(x)]'$$
$$= -e^{\cos(x)}\cdot \sin x$$

$\ast\ [\ln(x)]' = \frac{1}{x}$

$$[\sin(x^2\cdot\ln(x))]' = \cos(x^2\cdot\ln(x))\,[x^2\cdot\ln(x)]'$$
$$= \cos(x^2\cdot\ln(x))\,[2x\cdot\ln(x) + x]$$

$$h'(x) = e^{\cos(x)}[\sin(x^2\cdot\ln(x))]' + [e^{\cos(x)}]'\sin(x^2\cdot\ln(x))$$

e) One more proof (Please?)

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$\ln(x)$ is the inverse function of e^x.

$\rightarrow \frac{d}{dx} f^{-1} = \frac{1}{f'\circ f^{-1}}$ (inverse)

$$\ln(x) = f^{-1}, \quad f = e^x \Rightarrow f' = e^x$$

$$\frac{d}{dx}\ln(x) = \frac{1}{f'\circ f^{-1}} = \frac{1}{e^x\circ\ln(x)} = \frac{1}{x}$$

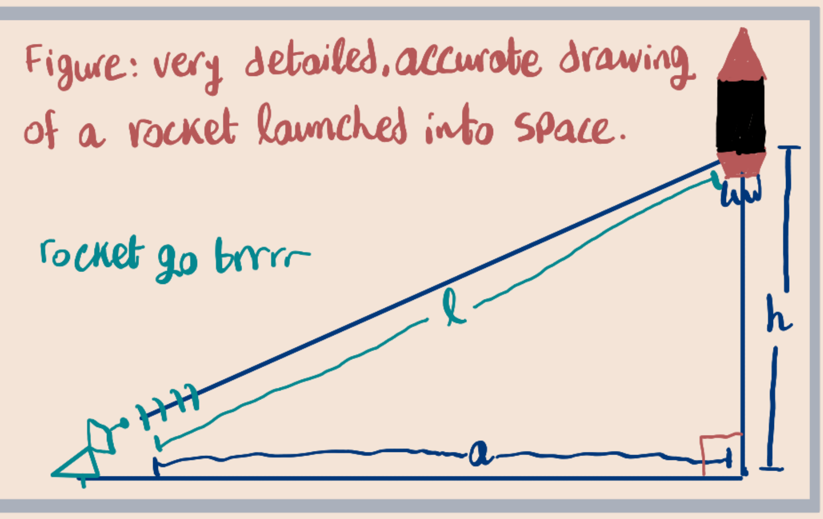
$$e^x\circ\ln(x) = e^{\ln(x)} = x \iff \ln(e^{\ln(x)}) = \ln(x)$$

$$\iff \ln(x)\cdot\ln(e) = \ln(x) \quad \div\ln(x) \iff \ln_e(e) = 1$$

$x \neq 1$

$:\iff e^1 = e$

2. Implicit Differentiation.



1. What is a relation between a, l, h?

$$a^2 + h^2 = l^2.$$

2. Can we find the rate of change of height with respect to time?

$\frac{d}{dt}$, $\frac{dh}{dt}$

$$\frac{d}{dt}\left(a^2 + h^2 = l^2\right)$$

$$= \cancel{2a\frac{da}{dt}}_{=0} + 2h\frac{dh}{dt} = 2l\frac{dl}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{l}{h}\frac{dl}{dt}$$

$* \; h = \sqrt{l^2 - a^2}$

$$\boxed{\Rightarrow \frac{dh}{dt} = \frac{l}{\sqrt{l^2 - a^2}} \cdot \frac{dl}{dt}}$$

3. Find this rate of change when

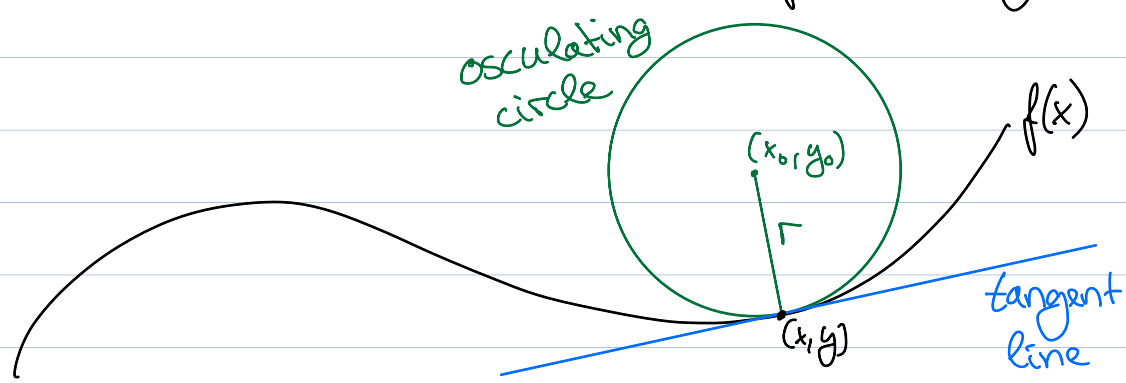
$\rightarrow a = 3\text{Km}$ $\quad l = 5\text{Km}$ $\quad \frac{dl}{dt} = 1000 \text{ m/s}.$

$$\Rightarrow \frac{dh}{dt} = \frac{l}{\sqrt{l^2 - a^2}} \cdot \frac{dl}{dt}$$

$$\boxed{= \frac{5\,\cancel{\text{Km}}}{4\,\cancel{\text{Km}}} \cdot 1000 \text{ m/s} = 1250 \text{ m/s}}$$

3. Application of 2^{nd} Derivative.

Osculating circle:
- touches graph of f at point (x,y)
- it has the same first and second derivative as f at (x,y)



"osculating circle provides second order local approximation to graph of f, while tangent line is only first-order approximation."

Q: What is radius r of osculating circle?

(a) $\boxed{(x-x_0)^2 + (y-y_0)^2 = r^2}$ eqⁿ circle.

1. (x_0, y_0) is the center of ○. They are constants.

Now apply $\frac{d}{dx}$

$$\frac{d}{dx}\left[(x-x_0)^2 + (y-y_0)^2 = \underbrace{r^2}_{\text{constant}}\right]$$

$$\Leftrightarrow \text{(b)} \quad \frac{\cancel{2}}{\cancel{2}}(x-x_0)\cdot \underbrace{\frac{dx}{dx}}_{=1} + \frac{\cancel{2}}{\cancel{2}}(y-y_0)\frac{dy}{dx} = \frac{\cancel{0}}{\cancel{2}}$$

$$\Leftrightarrow (x-x_0) + (y-y_0)\frac{dy}{dx} = 0 \Leftrightarrow \boxed{-(y-y_0)\cdot\frac{dy}{dx} = x - x_0}$$

Apply $\frac{d}{dx}$ again!

$$\Leftrightarrow \frac{d}{dx}\left[(x-x_0) + (y-y_0)\frac{dy}{dx} = 0\right]$$

$$\Leftrightarrow 1 + \left(\frac{d}{dx}\left[(y-y_0)\frac{dy}{dx}\right] = \left(\frac{dy}{dx}\right)^2 + (y-y_0)\underbrace{\frac{d}{dx}\cdot\frac{dy}{dx}}_{=\frac{d^2y}{dx^2}}\right) = 0$$

$$\Leftrightarrow (y-y_0)\frac{d^2y}{dx^2} = -1 - \left(\frac{dy}{dx}\right)^2 \Leftrightarrow \boxed{\frac{-1-\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = y - y_0}$$

Now define this magic circle with the following Properties:

0. y is my circle

1. $y' = f'(x)$

2. $\frac{d^2y}{dx^2} = f''(x)$

$\Leftrightarrow$ "my magic circle approximates f up to the 2nd derivative"

$$\boxed{\frac{-1-\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = y - y_0 = \frac{-1-(f'(x))^2}{f''(x)}} \quad \text{(c)}$$

$$\boxed{-(y-y_0)\cdot\frac{dy}{dx} = x - x_0 = \frac{1+(f'(x))^2}{f''(x)}\cdot f'(x)} \quad \text{(d)}$$

Plug (c), (d) in (a)

$$(x-x_0)^2+(y-y_0)^2=r^2$$

$$\left(\frac{1+(f'(x))^2}{f''(x)}\cdot f'(x)\right)^2+\left(\frac{1+(f'(x))^2}{f''(x)}\right)^2=r^2$$

$$\left(\frac{1+(f'(x))^2}{f''(x)}\right)^2(f'(x))^2+\left(\frac{1+(f'(x))^2}{f''(x)}\right)^2=r^2$$

$$\frac{\left(1+(f'(x))^2\right)^2}{(f''(x))^2}\cdot\left((f'(x))^2+1\right)^1=r^2$$

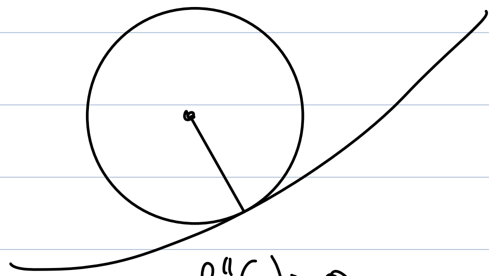
$$\frac{\left(1+(f'(x))^2\right)^3}{(f''(x))^2}=r^2 \Leftrightarrow \sqrt{\frac{\left(1+(f'(x))^2\right)^{\cancel{3}\,3/2}}{(f''(x))^{\cancel{2}}}}=r$$

$$\Leftrightarrow r=\frac{\left(1+(f'(x))^2\right)^{3/2}}{f''(x)},\quad \text{define } K=\frac{1}{r}\Rightarrow K=\frac{\overbrace{f''(x)}^{\text{controls the sign.}}}{\underbrace{\left(1+(f'(x))^2\right)^{3/2}}_{>0}}$$

$\boxed{\text{conclusion:}}$

+ve Curvature $= f''(x)>0$

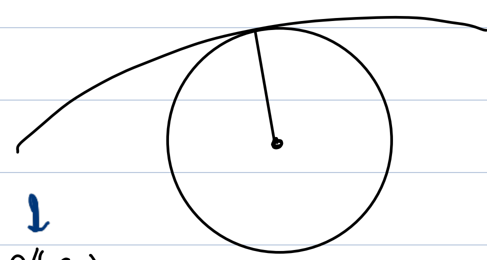
−ve Curvature $= f''(x)<0$



$f''(x) > 0$

$K > 0$

(circle above)



$f''(x) < 0$

$K < 0$

(circle below)