Agenda

1. Systems of linear equations

2. Solutions & Geometric Interpretation

Bonus. The Derivative Operator

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1. Systems of linear equations.

The fundamental goal of linear algebra is to solve systems of linear equations.

$$\boxed{\begin{aligned} 10x_1 + 6x_2 &= 1 \\ 9x_1 + 5x_2 &= 1 \end{aligned}}$$

Exercise. Find x_2. Hint. "Eliminate" one variable

$$\boxed{\frac{-9}{10} \times 10 = -9}$$

$$\begin{aligned} &\times \tfrac{-9}{10} \quad 10x_1 + 6x_2 = 1 \\ \hline &-9x_1 - \frac{27}{5} = -\frac{9}{10} \\ &\oplus \quad 9x_1 + 5x_2 = 1 \\ \hline &\overset{=0}{\cancel{0}}x_1 + -\frac{2}{5}x_2 = \frac{1}{\cancel{10}\,2} \quad \times -\frac{\cancel{5}}{2} \end{aligned}$$

$$\boxed{\Rightarrow x_2 = \frac{-1}{4}}$$

We can formulate this system using the language of matrices.

$$\underbrace{\begin{pmatrix} 10 & 6 \\ 9 & 5 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} = x_1 \begin{pmatrix} 10 \\ 9 \end{pmatrix} + x_2 \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 10x_1 + 6x_2 \\ 9x_1 + 5x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{b} \Rightarrow \begin{aligned} 10x_1 + 6x_2 &= 1 \\ 9x_1 + 5x_2 &= 1 \end{aligned}$$

We write

coefficient matrix

$Ax = b$. solution vector

vector of variables.

Exercise. Express $\left\{ \begin{aligned} 3x + 2y + z &= 24 \\ 4y - z &= 35 \\ x &= -3 \end{aligned} \right.$ in matrix form.

$$\boxed{Ax = b \iff \begin{array}{ccc} x & y & z \\ \end{array} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24 \\ 35 \\ -3 \end{pmatrix}}$$

Here is how we proceed.

Augmented Matrix.

- Glue A to b

$$\left(\begin{array}{cc|c} x_1 & x_2 & \overset{=}{\uparrow} \\ 10 & 6 & 1 \\ 9 & 5 & 1 \end{array}\right) \Leftrightarrow \begin{array}{l} 10x_1 + 6x_2 = 1 \\ 9x_1 + 5x_2 = 1 \end{array}$$

Goal: simplify to

$$\left(\begin{array}{cc|c} \boxed{1} & 0 & x_1 \\ 0 & \boxed{1} & x_2 \end{array}\right) \overset{\substack{\text{reading} \\ \text{off...} \\ \downarrow}}{\Leftrightarrow} \begin{array}{l} x_1 = x_1 \\ x_2 = x_2 \end{array} \qquad \boxed{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}$$

Ones on the diagonal & zero everywhere else!

Question: how?

$$\left(\begin{array}{cc|c} 10 & 6 & 1 \\ 9 & 5 & 1 \end{array}\right) \xrightarrow{\text{Path}} \left(\begin{array}{cc|c} 1 & 0 & x_1 \\ 0 & 1 & x_2 \end{array}\right)$$

Answer.... Gaussian Elimination.

Rules,

1. Multiply by a constant,

$$\begin{array}{l} 10x_1 + 6x_2 = 1 \\ 9x_1 + 5x_2 = 1 \Leftrightarrow 18x_1 + 10x_2 = 2 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{cc|c} 10 & 6 & 1 \\ 9 & 5 & 1 \end{array}\right)_{\rightsquigarrow \times 2} = \left(\begin{array}{cc|c} 10 & 6 & 1 \\ 18 & 10 & 2 \end{array}\right)$$

2. Add equations,

$$\left\{\begin{array}{l} \quad 10x_1 + 6x_2 = 1 \\ \oplus\ 9x_1 + 5x_2 = 1 \\ \hline 19x_1 + 11x_2 = 2 \end{array}\right. \Leftrightarrow \begin{array}{l} 10x_1 + 6x_2 = 1 \\ 19x_1 + 11x_2 = 2 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{cc|c} 10 & 6 & 1 \\ 9 & 5 & 1 \end{array}\right)_{\rightsquigarrow +R_1} = \left(\begin{array}{cc|c} 10 & 6 & 1 \\ 19 & 11 & 2 \end{array}\right).$$

Earlier we did ...

$$
\begin{array}{rl}
\times -\frac{9}{10} & 10x_1 + 6x_2 = 1 \\
\hline
 & -9x_1 - \frac{27}{5} = -\frac{9}{10} \\
\oplus & 9x_1 + 5x_2 = 1 \\
\hline
 & \overset{=0}{\cancel{0}}x_1 + -\frac{2}{5}x_2 = \frac{1}{10}2 \quad \times -\frac{5}{2}
\end{array}
$$

$$\boxed{\Rightarrow x_2 = \frac{-1}{4}}$$

Exercise. Use augmented matrix notation!

$$
\left(\begin{array}{cc|c} 10 & 6 & 1 \\ 9 & 5 & 1 \end{array}\right) \xrightarrow{+ -\frac{9}{10}R_1} = \left(\begin{array}{cc|c} 10 & 6 & 1 \\ 0 & -\frac{2}{5} & \frac{1}{10} \end{array}\right) \xrightarrow{\times -\frac{5}{2}} = \left(\begin{array}{cc|c} 10 & 6 & 1 \\ 0 & 1 & -\frac{1}{4} \end{array}\right)
$$

Exercise. Solve using Gaussian Elimination!

$$
\left(\begin{array}{c:c|c} 10 & 6 & 1 \\ 0 & 1 & -\frac{1}{4} \end{array}\right) \xrightarrow{+ -6R_2} = \left(\begin{array}{c:c|c} 10 & 0 & \frac{5}{2} \\ 0 & 1 & -\frac{1}{4} \end{array}\right) \xrightarrow{\times \frac{1}{10}} = \left(\begin{array}{c:c|c} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} \end{array}\right)
$$

$$
\begin{array}{rc:c|c}
\times -6 & 0 & 1 & -\frac{1}{4} \\
\hline
 & 0 & -6 & \frac{3}{2} \\
\oplus & 10 & 6 & 1 \\
\hline
 & 10 & 0 & 5/2
\end{array}
$$

read off...

$$\Leftrightarrow \boxed{\begin{array}{l} x_1 = \frac{1}{4} \\ x_2 = -\frac{1}{4} \end{array}}$$

Exercise. Find x_1, x_2 for the system

$$
\begin{array}{cc|c} x_1 & x_2 & \\ \hline 1 & 2 & 10 \\ 3 & 4 & 15 \end{array}
$$

First, let us make zeroes in the first column.

$$
\left(\begin{array}{cc|c} \boxed{1} & 2 & 10 \\ \textcircled{3} & 4 & 15 \end{array}\right) \times -3 = \left(\begin{array}{c:c|c} \boxed{1} & 2 & 10 \\ 0 & -2 & -15 \end{array}\right) \times 1
$$

Next Column...

$$
= \left(\begin{array}{c:c|c} \boxed{1} & \textcircled{0} & -5 \\ 0 & \boxed{-2} & -15 \end{array}\right) \times -\frac{1}{2}
$$

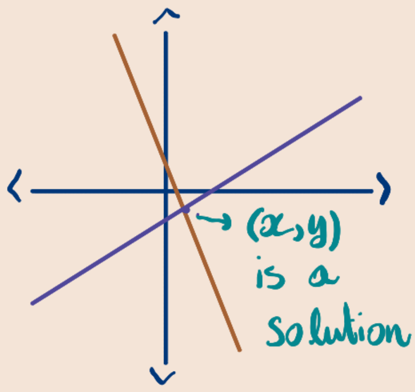
$$
= \left(\begin{array}{c:c|c} \boxed{1} & 0 & -5 \\ 0 & \boxed{1} & \frac{15}{2} \end{array}\right) \Leftrightarrow \boxed{\begin{array}{l} x_1 = -5 \\ x_2 = \frac{15}{2} \end{array}}
$$

2. Solutions & Geometric Interpretation.

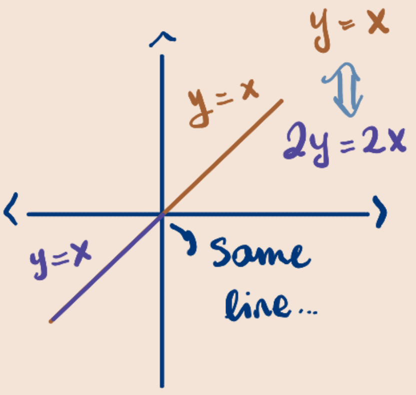
Let us consider the visually feasible two-dimensional plane.

Example.
$$\begin{cases} a_1x+b_1y=c_1 \iff y_1=m_1x+c_1 \\ a_2x+b_2y=c_2 \iff y_2=m_2x+c_2 \end{cases}$$

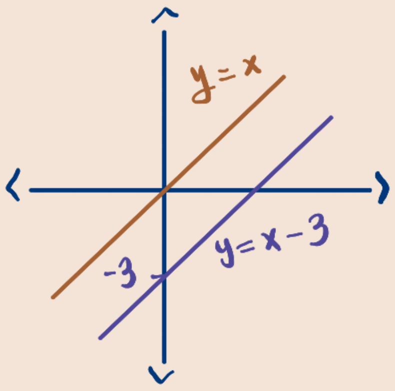
Intersect once! | **Intersect always...** | **Never Intersect.**



One solution (x,y)



Infinitely many solutions



no Solutions (x,y)

Let us consider two equations with the same data.

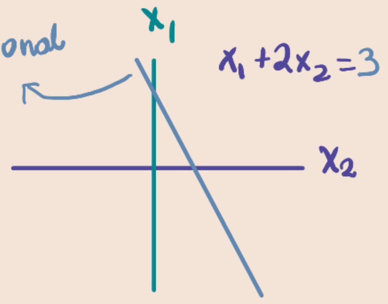
Example.
$$\begin{cases} x_1+2x_2=3 \\ 4x_1+8x_2=12 \end{cases} \iff \left(\begin{array}{cc|c} x_1 & x_2 & \\ \boxed{1} & 2 & 3 \\ \textcircled{4} & 8 & 12 \end{array}\right) \times -4$$

↓ write the system

the second equation has no data...
$$= \left(\begin{array}{cc|c} 1 & 2 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \iff \begin{cases} x_1+2x_2=3 \\ \cancel{0x_1+0x_2=0} \end{cases}$$

→ zeroes!

Solution is a one-dimensional space



In this case, we move the extra variable to the other side.

$$\left(\begin{array}{cc|c} x_1 & x_2 & \\ 1 & 2 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) = \left(\begin{array}{cc|c} x_1 & x_2 & = \\ 1 & 2 & 3 \end{array}\right) = \left(\begin{array}{c|cc} x_1 & = & x_2 \\ 1 & -2 & 3 \end{array}\right).$$

Finally, write

$$\begin{pmatrix} x_1 \\ \boxed{x_2} \end{pmatrix} = \begin{pmatrix} -2x_2+3 \\ \boxed{x_2} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad x_2 \in \mathbb{R}$$

$$\text{Variables} - \underbrace{\text{Equations}}_{\text{lin independent}} \begin{cases} +\text{ve} \Rightarrow \infty \text{ solutions} \\ 0 \Rightarrow \text{one solution} \\ -\text{ve} \Rightarrow \text{no solutions. overconstrained!} \end{cases}$$

Example.

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 1 & 2 & 3 & 4 & 5 & 2 \\ 23 & 41 & 7 & 2 & 4 & 4 \end{array} \quad \begin{cases} \text{Equations} = 2 \\ \text{Variables} = 5 \end{cases}$$

* Var - Eq$^{\underline{n}}$ = 3 = dimension of solution space.

$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & 3 & 4 & 5 & 2 \\ 23 & 41 & 7 & 2 & 4 & 4 \end{array}\right) \times -23 = \left(\begin{array}{c:cccc|c} \boxed{1} & 2 & 3 & 4 & 5 & 2 \\ 0 & \boxed{-5} & -62 & -90 & -111 & -42 \end{array}\right) \times \frac{2}{5}$$

$$= \left(\begin{array}{c:cccc|c} \boxed{1} & 0 & -\frac{111}{5} & -32 & -\frac{197}{5} & -\frac{76}{5} \\ 0 & \boxed{-5} & -62 & -90 & -111 & -42 \end{array}\right) \times -\frac{1}{5} = \left(\begin{array}{cc:ccc|c} \boxed{1} & 0 & -\frac{111}{5} & -32 & -\frac{197}{5} & -\frac{76}{5} \\ 0 & \boxed{1} & \frac{62}{5} & 18 & \frac{111}{5} & \frac{42}{5} \end{array}\right)$$

Draft.

$$\begin{array}{c:cccc|c} \times\frac{2}{5} \quad 0 & -5 & -62 & -90 & -111 & -42 \\ \hline 0 & -2 & -\frac{124}{5} & -36 & -\frac{222}{5} & -\frac{86}{5} \\ \oplus \quad 1 & 2 & 3 & 4 & 5 & 2 \\ \hline 1 & 0 & -\frac{111}{5} & -32 & -\frac{197}{5} & -\frac{76}{5} \end{array}$$

The next step is to move with a change of sign.

$$= \left(\begin{array}{cc|cccc} \boxed{1} & 0 & +\frac{111}{5} & +32 & +\frac{197}{5} & -\frac{76}{5} \\ 0 & \boxed{1} & -\frac{62}{5} & -18 & -\frac{111}{5} & \frac{42}{5} \end{array}\right)$$

To conclude, write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{111}{5}x_3 + \frac{32}{5}x_4 + \frac{197}{5}x_5 + \frac{76}{25} \\ -\frac{62}{5}x_3 - 18x_4 - \frac{111}{5}x_5 + \frac{42}{5} \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{76}{25} \\ \frac{42}{5} \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \frac{111}{25} \\ -\frac{62}{5} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{32}{5} \\ -18 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} \frac{197}{5} \\ -\frac{111}{5} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R}$$

$$\begin{cases} x = 2 \\ x = 3 \\ x = 4 \end{cases} \Leftrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \Leftrightarrow \left(\begin{array}{c|c} \boxed{1} & 2 \\ 1 & 3 \\ 1 & 4 \end{array}\right) \begin{array}{l} \times -1 \\ \times -1 \end{array} = \left(\begin{array}{c|c} \boxed{1} & 2 \\ 0 & -1 \\ 0 & -2 \end{array}\right)$$

$$\Leftrightarrow 0x = 0 = -1$$

$$\Leftrightarrow 0x = 0 = -2$$

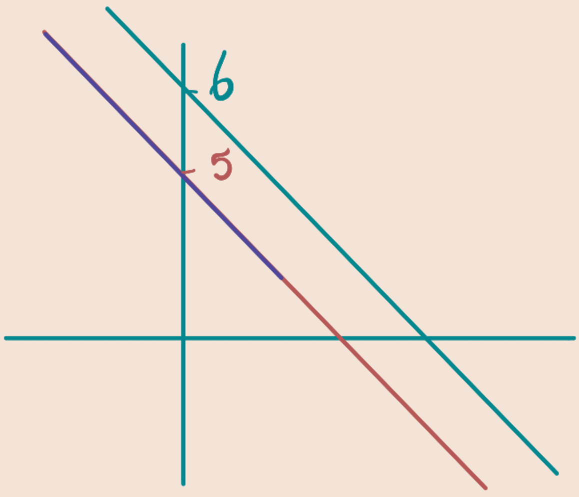
Contradiction....

Simpler Illustration.

$x+y=5$

~~$2x+2y=10$~~

$x+y=6$



$$\begin{array}{r} x+y=5 \\ \ominus \quad x+y=6 \\ \hline \boxed{0=-1} \end{array}$$

Let A be an $n \times m$ matrix. Then

$$\underbrace{\text{Rank}(A)}_{\substack{\text{How many dimensions} \\ \text{do the equations span?}}} + \overbrace{\text{Nullity}(A)}^{\substack{\text{if } \vec{b}=0\text{, what is} \\ \text{the dimension of} \\ \text{the solution space?}}} = m. \quad \text{number of columns}$$

Example. $\left(\begin{array}{cc:ccc|c} \boxed{1} & 0 & -\frac{111}{5} & -32 & -\frac{197}{5} & \boxed{0} \\ 0 & \boxed{1} & \frac{62}{5} & 18 & \frac{111}{5} & 0 \end{array}\right)$ has rank 2. The two equations (rows) are linearly independent. The solution space

$$x_3 \begin{pmatrix} \frac{111}{25} \\ -\frac{62}{5} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{32}{5} \\ -18 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} \frac{197}{5} \\ -\frac{111}{5} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R}$$

has three linearly independent vectors, hence Nullity $= 3$.

$$\text{Rank} + \text{Nullity} = 2 + 3 = 5 = \text{number of columns}$$

Example. $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ has Rank $= 1$. The two vectors span a line. Similarly, Nullity $= 1$, as the solution

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 4 & 8 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad x_2 \in \mathbb{R}$$

indicates. $(1+1=2.)$

Some matrices have full rank.

Exercise. Find rank & nullity of

$$\begin{pmatrix} 10 & 6 \\ 9 & 5 \end{pmatrix}.$$

Rank. The span of $\begin{pmatrix} 10 \\ 9 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ is two-dimensional.

Nullity. Solution to $A\mathbf{x} = 0$ is

$$\left(\begin{array}{cc|c} 10 & 6 & 0 \\ 9 & 5 & 0 \end{array}\right) = \cdots = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \Rightarrow \boxed{\begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}}$$

Zero-dimensional.

$$\text{Rank} + \text{Nullity} = 2 + 0 = 2 = \text{number of columns}$$

Bonus. The Derivative Operator

$V = P_2(\mathbb{R}) = \{ p(x) : p(x) \text{ is quadratic} \}$ is the vector space.

1. Familiarize yourself with examples.

$$\hookrightarrow 2x^2 + 4 \overset{\text{in standard basis}}{=} 2 \cdot x^2 + 0 \cdot x + 4 \cdot 1$$

More generally,

$$\hookrightarrow ax^2 + bx + c \cdot 1 , \quad a, b, c \in \mathbb{R}$$

2. What does taking the derivative do to the coefficients?

$$D : p(x) = ax^2 + bx + c \cdot 1 \xmapsto[\text{to}]{\text{maps}} p'(x) = 0 \cdot x^2 + 2a \cdot x + b \cdot 1$$

3. Is D a linear operator? yes....

Operator

check this....

Def (Linear Map.) A map $\ell : V \to W$ is linear if

(i) $\ell(v_1 + v_2) = \ell(v_1) + \ell(v_2)$ $\quad \forall v_1, v_2 \in V$

(ii) $\ell(\lambda v) = \lambda \ell(v)$ $\quad \forall v \in V, \lambda \in \mathbb{F}$

$$\underbrace{\begin{array}{ccc} x^2 & x & 1 \\ \hline a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}}_{= D} \cdot \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{= p(x)} = \underbrace{\begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix}}_{= p'(x)}$$

D is the matrix (= linear operator) that sends $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix}$. Can you explicitly compute it?

Hint. Proceed with matrix multiplication, & equate both sides.