1. Calculate $\int_0^1 \frac{1}{x^2+14x+98}\,dx = \int_{0}^{1=x} \frac{dx}{(x+7)^2+49} = I$

" x

- $x^2+14x+98 = x^2+2\cdot 7\cdot x+\underbrace{7^2}_{=49}+49 = (x+7)^2+49$
- $u = x+7 \Rightarrow du = dx$

$$I = \int_{u=7}^{u=8} \frac{1}{(u)^2+49}\,du = \int_{u=7}^{u=8} \frac{1\cdot\frac{1}{49}}{\left((u)^2+\cancel{49}^{1}\right)\cdot\cancel{\frac{1}{49}}}\,du$$

$$\boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}}$$

$$= \frac{1}{7}\int_{u=7}^{u=8} \frac{1}{7}\cdot\frac{1}{\left(\frac{u}{7}\right)^2+1}\,du \overset{(*)}{=} \frac{1}{7}\left(\tan^{-1}\frac{u}{7}\Big|_{u=7}^{u=8} = \tan^{-1}\frac{8}{7} - \tan^{-1}\frac{7}{7}\right)$$

$$= \frac{1}{7}\left(\tan^{-1}\frac{8}{7} - \frac{\pi}{4}\right) \quad \boxed{/\!/}$$

(*) $\left(\tan^{-1}\frac{u}{7}\right)' = \frac{1}{7}\cdot\frac{1}{1+\left(\frac{u}{7}\right)^2}$

Or...

$$I = \int_{u=7}^{u=8} \frac{1}{\underbrace{(u)^2+49}_{\text{want: } 1}}\,du = \int_{u=7}^{u=8} \frac{1\cdot\frac{1}{49}}{\left(\cancel{(u)}^{\frac{u}{7}}{}^2+\cancel{49}^{1}\right)\cdot\cancel{\frac{1}{49}}}\,du$$

$$\boxed{(*)\ \tan^2\theta + 1 = \sec^2\theta}$$

$$\boxed{(**)\ (\tan\theta)' = \sec^2\theta}$$

$$= \frac{1}{49}\int_{u=7}^{u=8} \frac{1}{\left(\frac{u}{7}\right)^2+1}\,du, \quad \frac{1}{7}u = \tan\theta \Rightarrow du \overset{(**)}{=} 7\sec^2\theta\, d\theta$$

$$\boxed{(***)\quad \frac{1}{7}u = \tan\theta \Rightarrow \tan^{-1}\left|\frac{u}{7}\right| = \theta}$$

$$= \frac{1}{49}\int_{u=7}^{u=8} \frac{7\sec^2\theta}{(\tan\theta)^2+1}\,d\theta \overset{(*)}{=} \frac{1}{\cancel{49}\,7}\int_{u=7}^{u=8} \frac{\cancel{7\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$$

$$= \frac{1}{7}\int_{u=7}^{u=8} 1\,d\theta = \frac{1}{7}\cdot\left(\theta\Big|_{u=7}^{u=8}\right) \overset{(***)}{=} \frac{1}{7}\cdot\left(\tan^{-1}\frac{u}{7}\Big|_{u=7}^{u=8}\right)$$

$$= \frac{1}{7}\cdot\left(\tan^{-1}\frac{8}{7} - \frac{\pi}{4}\right) \quad \boxed{/\!/}$$

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2. Let $P(t)$ denote the number of bacteria in a sample at time t. Initially, $P(0) = 100$ and increases at a rate $\frac{dP}{dt} = 20e^{3t}$. What is the population at $t=50$?

Goal: $P(50) \leftarrow P(t)$

Have: $P(0) = 100$, $\cancel{\frac{dP}{dt} = 20e^{3t}} \Rightarrow$ " $\int \frac{dP}{\cancel{dt}}(t)\,\cancel{dt}$ " $= P(t)$

$$\boxed{(f\circ g)' = (f'\circ g)\cdot(g')}$$

$20\int e^{3t}\,dt = 20\cdot\frac{1}{3}e^{3t} + C = P(t)$.

$100 = P(0) = 20\cdot\frac{1}{3}\underbrace{e^{3\cdot 0}}_{=1} + C = 20\cdot\frac{1}{3} + C \Rightarrow C = \frac{280}{3}$

$\Rightarrow P(t) = 20\cdot\frac{1}{3}e^{3t} + \frac{280}{3} \Rightarrow P(50) = \frac{20}{3}\cdot e^{3\cdot 50} + \frac{280}{3}$

$$\boxed{\approx 9.3\times 10^{65}}$$

3. $\int_2^1 \frac{2y^3-6y^2}{y^2}\,dy \overset{(*)}{=} \int_2^1 (2y-6)\,dy = y^2-6y\Big|_2^1.$

$(*)\ \frac{2y^{\cancel{3}}-6\cancel{y^2}}{\cancel{y^2}} = 2y-6$

$= (1^2-6\cdot 1)-(2^2-6\cdot 2) = 3$

More generally, $f(x)\Big|_a^b = f(b)-f(b).$

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4. $\int_0^{1/2} \frac{2x^2+2}{x^2-1}\,dx$

$= I \overset{(*)}{=} 2\int_0^{\frac{1}{2}} 1\,dx + I_1$

$\boxed{= 1 - 2\cdot \ln(3)}$

$(*)\ \frac{2x^2+2}{x^2-1} = 2\,\frac{x^2+1}{x^2-1} = 2\cdot\frac{x^2\cancel{-1}+2}{\cancel{x^2-1}} = 2+\frac{4}{x^2-1}$

$(*)\ \frac{2x^2+2}{x^2-1} = \quad x^2-1 \overline{\big)\ 2x^2+2}$ $2,\ r=4$ $= 2+\frac{4}{x^2-1}$

$-(\cancel{2x^2}-2)$

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$I_1 = \int_0^{\frac{1}{2}} \frac{4}{x^2-1}\,dx\ ,\quad \frac{4}{x^2-1} = \frac{4}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1} = \frac{2}{x-1}-\frac{2}{x+1}.$

$\boxed{A = \frac{4}{(1+1)} = \frac{4}{2} = 2\ ,\quad B = \frac{4}{-1-1} = \frac{4}{-2} = -2}$

$I_1 = 2\left(\int_0^{\frac{1}{2}} \frac{1}{x-1}\,dx - \int_0^{\frac{1}{2}} \frac{1}{x+1}\,dx\right) = 2\left(\ln|x-1| - \ln|x+1|\Big|_0^{\frac{1}{2}}\right)$ $\boxed{(\ln x)' = \frac{1}{x}}$

$= 2\left((\ln\tfrac{1}{2} - \ln\tfrac{3}{2}) - (\underbrace{\cancel{\ln(1)}}_{=0} - \underbrace{\cancel{\ln(1)}}_{=0})\right) = \ln(\tfrac{1}{3})\cdot 2$

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5. $\int_0^1 \frac{3x^2+12x+11}{(x+1)(x+2)(x+3)}\,dx = I$

$\frac{3x^2+12x+11}{(x+1)(x+2)(x+3)} = \frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}$

$\boxed{A = \frac{3(-1)^2+12(-1)+11}{(-1+2)(-1+3)} = \frac{2}{2} = 1,\quad B = \frac{3(-2)^2+12(-2)+11}{(-2+1)(-2+3)} = \frac{-1}{-1} = 1}$

$I = \int_0^1 \frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}\,dx = \ln|x+1|+\ln|x+2|+\ln|x+3|\Big|_0^1$

$= \ln|(x+1)(x+2)(x+3)|\Big|_0^1 = \ln(2\cdot3\cdot4) - \ln(1\cdot2\cdot3)$

$\boxed{(\ln(x))' = \frac{1}{x}}$

$\boxed{2\ln 2} = \ln 2^2 = \ln\frac{(\cancel{2}\cdot\cancel{3}\cdot 4)}{(1\cdot\cancel{2}\cdot\cancel{3})} = \ln 4$

6. $\int \frac{e^x}{e^{2x}-e^x}\,dx$, $e^{2x}=(e^x)^2$

$\boxed{(\ln x)' = \frac{1}{x}}$

$$\int \frac{e^x}{(e^x)^2-e^x}\,dx \text{ , } u=e^x \Rightarrow du = e^x\,dx \Rightarrow dx = \frac{du}{e^x}$$

$$\int \frac{\cancel{e^x}}{(u)^2-u}\frac{du}{\cancel{e^x}} = \int \frac{1}{u^2-u}\,du = \int \frac{1}{u(u-1)}\,du = \int \frac{-1}{u}+\frac{1}{u-1}\,du$$

$$\boxed{\frac{1}{u(u-1)} = \frac{A}{u}+\frac{B}{u-1} \text{ . } A = \frac{1}{(0-1)} = -1 \text{ , } B = \frac{1}{1} = 1}$$

$$= \ln|u-1| - \ln|u| + C = \ln\left(\frac{|u-1|}{|u|}\right) + C = \boxed{\ln\left(\left|\frac{e^x-1}{e^x}\right|\right)+C}$$

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7. $\frac{dy}{dx} = \sqrt{\frac{(1-\cos x \sin x)^2}{\cos^4 x}e^{-2x}-1}$, find the length L of $y(x)$ on $x \in [0,1]$ using $L = \int_{P_0}^{P_1} ds = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. Hint. Recall $(\tan x)'$.
$\qquad = \sec^2 x$

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{\cancel{1}+\left(\cancel{\sqrt{\frac{(1-\cos x\sin x)^2}{\cos^4 x}e^{-2x}\cancel{-1}}}\right)^{\cancel{2}}}\,dx$$

$\boxed{\left(\frac{f}{g}\right)' = \frac{gf'-fg'}{g^2}}$

$$= \int_0^1 \frac{(1-\cos x\sin x)}{\cos^2 x}e^{-x}\,dx = \int_0^1 \frac{\sec^2 x - \tan x}{e^x}\,dx$$

$$= \int_0^1 \frac{(\tan x)'-\tan x}{e^x}\,dx = \int_0^1 \frac{e^x(\tan x)' - \tan x\,(e^x)'}{(e^x)^2}\,dx$$

$$= \frac{\tan x}{e^x}\Big|_0^1 = \frac{\tan 1}{e} - \cancel{\frac{\overset{=0}{\tan 0}}{1}} = \boxed{\frac{\tan 1}{e}}$$

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8. $f(x) = x\left(\frac{e^x - e^{-x}}{2}\right)\tan x \Rightarrow \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_0^1 f(x)\,dx$

(*) Claim. $f(-x) = -f(x) \Leftrightarrow$: $f(x)$ is odd.

$\boxed{-\frac{\sin(x)}{\cos(x)} = \tan(-x) = -\tan(x)}$

Pf.

$$f(-x) = -x\left(\frac{e^{-x}-e^{--x}}{2}\right)\tan -x = x\left(\frac{e^x - e^{-x}}{2}\right)(-\tan x) = -f(x) \quad \blacksquare$$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_0^1 f(x)\,dx = \int_0^{-1} -f(x)\,dx + \int_0^1 f(x)\,dx$$

$$\overset{(*)}{=} \int_{0=x}^{-1=x} f(\overset{1}{-x})\,dx + \int_0^1 f(x)\,dx$$

$$= \cancel{\int_0^1 f(x)\,dx} + \cancel{\int_0^1 f(x)\,dx} = 0$$

9. $\int_1^e \frac{\ln x}{x}\,dx = ?$ $\quad u = \ln(x) \Rightarrow du = \frac{1}{x}\,dx \Rightarrow dx = x \cdot du$

$$I = \int_{x=1}^{x=e} u \cdot \frac{1}{\cancel{x}}\,\cancel{x} \cdot du = \int_{u=\ln(1)}^{u=\ln(e)} u\,du = \frac{u^2}{2}\Big|_{u=0}^{u=1} = \frac{1^2}{2} - \cancel{\frac{0^2}{2}}^{=0}$$

$$\boxed{= \frac{1}{2}}$$

$$\boxed{\log_b k = n \Leftrightarrow b^n = k}$$

$\log_e e = \ln e = n \Leftrightarrow e^n = e^1 \Leftrightarrow 1 = n$

$\ln 1 = \log_e 1 = n \Leftrightarrow e^n = 1 \Leftrightarrow n = 0$

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10. $\int_0^{\pi/2} x \sin x \cos x\,dx = \frac{1}{2}\int_0^{\pi/2} x \sin(2x)\,dx$

$$\boxed{\sin(2x) = 2\sin x \cos x}$$

$$\boxed{= \frac{1}{2} \cdot \left(\frac{\pi}{4}\right) = \frac{\pi}{8}}$$

$$\boxed{\begin{array}{c} u = x \Rightarrow du = dx \\ dv = \sin 2x\,dx \Rightarrow \frac{-\cos 2x}{2} = v \end{array}}$$

$$I = \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du = x \cdot \frac{-\cos 2x}{2}\Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{-\cos 2x}{2} \cdot dx$$

$$= \frac{1}{2}\left(\int_0^{\pi/2} \cos 2x \cdot dx - x \cos 2x\Big|_0^{\pi/2}\right)$$

$$= \frac{1}{2}\left(\frac{\sin 2x}{2}\Big|_0^{\pi/2} - \left(\frac{\pi}{2}\underbrace{\cos \cancel{2} \cdot \frac{\pi}{\cancel{2}}}_{-1} - 0 \cdot \cancel{\cos 2 \cdot 0}^{=0}\right)\right)$$

$$= \frac{1}{2}\left(\left(\cancel{\frac{\sin 2 \cdot \frac{\pi}{2}}{2}}^{=0} - \cancel{\frac{\sin 2 \cdot 0}{2}}^{=0}\right) + \frac{\pi}{2}\right) = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$
