1. Which of the following does not have a horizontal asymptote?

a) $f(x) = \dfrac{\log|x^7|}{\log|x^3| + x}$, $\lim_{x \to \pm\infty} \dfrac{7\log|x|}{x} = 0$.

↳ x dominates $\log|x|$

$\boxed{\ln(x) / \log(x) \text{ grow slower than any polynomial}}$

b) $f(x) = \dfrac{\log|x|}{x}$, $\lim_{x \to \pm\infty} f(x) = 0$.

↳ x dominates $\log|x|$

c) $f(x) = \dfrac{a_0 + a_1 x + \dots + a_n x^n}{b_0 + b_1 x + \dots + b_n x^n}$, $a_i, b_i \neq 0$. $\lim_{x \to \pm\infty} \dfrac{a_n \cancel{x^n}}{b_n \cancel{x^n}} = \dfrac{a_n}{b_n}$.

$\boxed{e^x \text{ grows faster than any polynomial}}$

$\boxed{\text{d) } f(x) = \dfrac{e^{|x|}}{x^m + x^{m-1} + \dots + x + 1}, \ m \in \mathbb{N}}$ $\lim_{x \to \pm\infty} f(x) = +\infty$

2. Evaluate the limit $\lim_{x \to 0} \dfrac{12^x - 1}{x}$, $u = 12^x - 1 \xrightarrow{x \to 0} 0$. Then

(*) $u + 1 = 12^x \Rightarrow \ln(u+1) = \ln(12^x) = x\ln(12) \Rightarrow x = \dfrac{\ln(u+1)}{\ln(12)}$

$\lim_{u \to 0} u \cdot \dfrac{\ln(12)}{\ln(u+1)} = \ln(12) \lim_{u \to 0} \left(\dfrac{\cancel{u}}{\ln(u+1)} \cdot \dfrac{\cancel{1/u}}{\frac{1}{u}} = \dfrac{1}{\frac{1}{u}\ln(u+1)} = \dfrac{1}{\ln[(u+1)^{\frac{1}{u}}]} \right)$

$w = \dfrac{1}{u} \xrightarrow{u \to 0} \pm\infty \Rightarrow \lim_{w \to \infty} \dfrac{\ln(12) \cdot 1}{\ln[(1 + \frac{1}{w})^w]} = \lim_{w \to \infty} \left(\dfrac{\ln(12)}{\ln e} \right)$

$\boxed{(*) \ \ln(e) = 1 \Leftrightarrow e^1 = e}$ $\boxed{= \ln(12)}$

$$e = \lim_{w \to \infty} \left(1 + \frac{1}{w}\right)^w$$

---

3. Evaluate the limit $\lim_{N \to \infty} \left(\sum_{i=1}^{N} \dfrac{i^2}{N^3} = \dfrac{1}{N^3} \cdot \underbrace{\sum_{i=1}^{N} i^2}_{} \right) = L$

$\boxed{\sum_{i=1}^{N} i^2 = \dfrac{N \cdot (N+1)(2N+2)}{6}}$. $\Rightarrow L = \lim_{N \to \infty} \left(\dfrac{1}{N^3} \cdot \dfrac{N \cdot (N+1)(2N+2)}{6} \right.$

↳ Prove this using induction.

$\left(= \dfrac{N \cdot (N+1)(2N+2)}{6N^3} \right)$

$\boxed{L = \lim_{N \to \infty} \dfrac{2\cancel{N^3}}{6\cancel{N^3}} = \dfrac{1}{3}}$

4. Evaluate the limit $\lim_{N\to\infty}\sum_{i=1}^{N}\left(\frac{1}{i^2+i}=\frac{1}{i(i+1)}\right)=1$

Partial Fraction Decomposition: $\frac{1}{i(i+1)}=\frac{A}{i}+\frac{B}{i+1} \quad \times\, i(\cancel{i+1})$

$$\boxed{\frac{1}{i(i+1)}=\frac{1}{i}+\frac{-1}{i+1}}$$

$1 = A(i+1)+Bi$

$\quad = Ai + A + Bi$

$0\cdot i + 1 = (A+B)i + A \Rightarrow \begin{cases} \boxed{A=1} \\ A+B=0 \Rightarrow \boxed{B=-1} \end{cases}$

$$\boxed{\lim_{N\to\infty}(S_N=1)=1}$$

$\boxed{\text{Trick:}}$ $S_N=\sum_{i=1}^{N}\left(\frac{1}{i^2+i}=\frac{1}{i(i+1)}=\frac{1}{i}+\frac{-1}{i+1}\right)$ $\Rightarrow$ S_N Telescopic.

$$=\left(\sum_{i=1}^{N}\frac{1}{i}=1+\cancel{\sum_{i=1}^{N}\frac{1}{1+i}}\right)-\cancel{\sum_{i=1}^{N}\frac{1}{i+1}}=1$$

$\frac{1}{1}+\cancel{\frac{1}{2}}+\cancel{\frac{1}{3}}+\cancel{\dots}+\cancel{\frac{1}{N}}$ $\qquad$ $\cancel{\frac{1}{2}}+\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\dots+\cancel{\frac{1}{N}}$

5. Check by induction the correct statement.

a) ~~$\sum_{k=1}^{n}2^{k-1}=2^n+1$~~, $\boxed{n=1}$ $\sum_{k=1}^{1}2^{k-1}=2^{1-1}=2^0=1\neq 5=2^1+1$

$\boxed{\text{b)}}$ $\sum_{k=1}^{n}(2k-1)=n^2$ $\quad\boxed{n=1}$ $\sum_{k=1}^{1}(2k-1)=1=1$

c) ~~$n!<2^n$ for $n>4$~~ $\quad 5!\not<2^5$, $5!=120$, $2^5=32$

d) ~~$\sum_{k=1}^{n}k^3=n^2(n+1)^2$~~ $\quad 1\neq 4$

6. Let f be differentiable. Consider

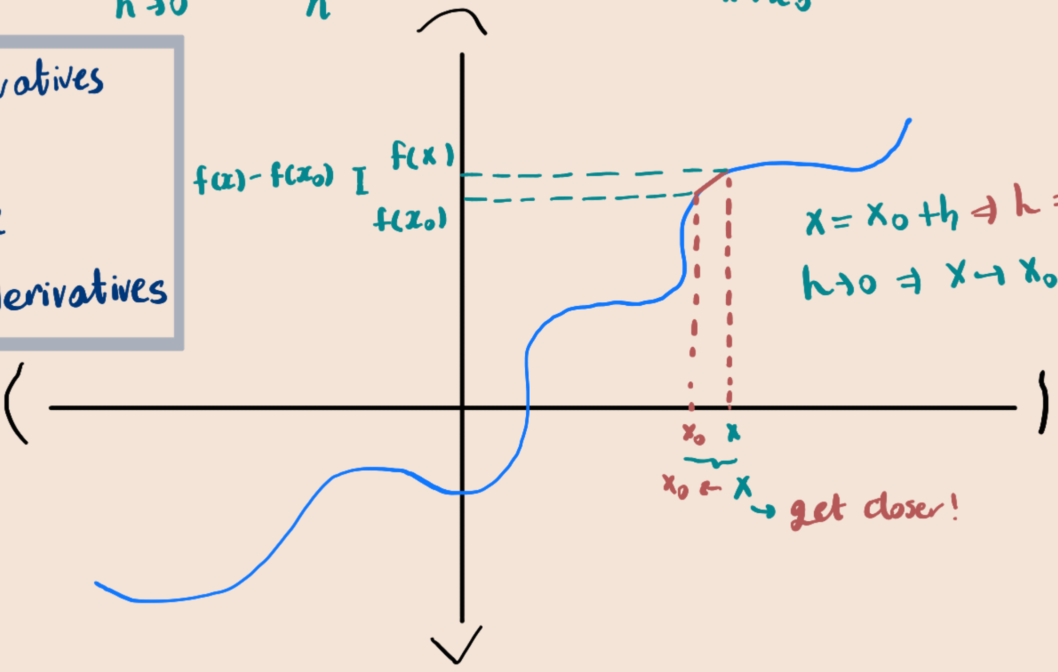
$$f_1(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \;;\; f_2(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}.$$

$\boxed{\text{a)}}$ f_1, f_2 derivatives

b) f_1 derivative

c) f_2 derivative

d) f_1, f_2 not derivatives



$x=x_0+h \Rightarrow h=x-x_0$

$h\to 0 \Rightarrow x\to x_0$

7. Let $f(x)=x^2+x$. Using the limit definition of a derivative, which of the following is the correct formulation?

$$\cdot\ f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},\quad f(x+h)=(x+h)^2+(x+h).$$

$$=\lim_{h\to 0}\frac{(x+h)^2+(x+h)-(x^2+x)}{h}$$

$$=\lim_{h\to 0}\frac{(x^2+2xh+h^2)+(x+h)-x^2-x}{h}$$

$$\Rightarrow \boxed{d)}\ f'(x)=\lim_{h\to 0}\frac{x^2+2xh+h^2+x+h-x^2-x}{h}$$

---

8. Let $m\geq 2$, and consider $f(x)=\begin{cases}x^m, & x\leq 0\\ 0, & x\geq 0\end{cases}$. Using the limit definition of a derivative, find $f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$.

$$f'_+(0)=\lim_{h\to 0^+}\left(\frac{f(h)}{h}=\frac{0}{h}=0\right)=0$$

$$\parallel$$

$$f'_-(0)=\lim_{h\to 0^-}\left(\frac{f(h)}{h}=\frac{h^{\cancel{m}}}{\cancel{h}}=h^{m-1}\right)=0$$

$$\boxed{f'(0)=0}$$

---

9. Consider $f(x)=\begin{cases}-x^2, & x<0\\ 0, & x=0\\ \sin x, & x>0\end{cases}$. Using the limit definition of a derivative, find $f'(0)$.

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}.$$

$$f'_+(0)=\lim_{h\to 0^+}\left(\frac{f(h)}{h}=\frac{\sin h}{h}\right)=1$$

$$f'_-(0)=\lim_{h\to 0^-}\left(\frac{f(h)}{h}=\frac{-h^{\cancel{2}}}{\cancel{h}}\right)=0$$

$$\cancel{f'(0)=\begin{cases}1\\0\end{cases}}\ \Rightarrow \text{no } f'(0).$$

10. The Rectified Linear Unit function $ReLU(x)$ is defined as $ReLU(x) := \max\{0,x\}$. Then, ReLU is differentiable...

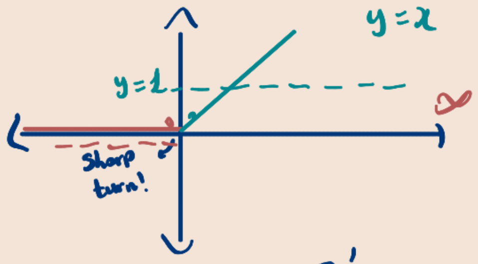
$\boxed{a)}$ everywhere but $x=0$, with $\dfrac{d}{dx} ReLU = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$.

$$ReLU(x) = \begin{cases} x & , x \geq 0 \\ 0 & , x \leq 0 \end{cases}$$

~~b) on finitely many points.~~

~~c) nowhere~~

~~d) everywhere.~~



$$\lim_{x \to 0^+} \left[ReLU(x)\right]' = 1$$

$$\lim_{x \to 0^-} \left[ReLU(x)\right]' = 0$$