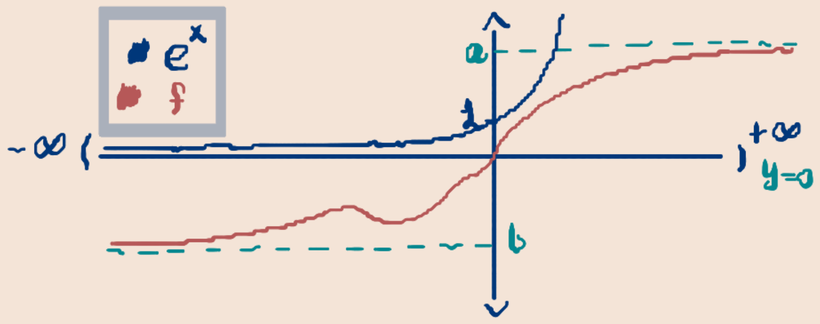
1. Find all horizontal asymptotes of $y(x) = \sin\left(\frac{1}{x}\right) \cdot x^2 + \frac{1}{x-2}$.

**Hint:** $\lim_{x \to 0} \frac{\sin x}{x} = 1$




**Ex**

$\lim_{x \to \infty} e^x = \infty$

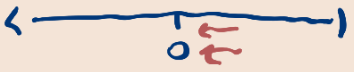
$\lim_{x \to -\infty} e^{-x} = 0$

- a, b are horizontal asymptotes of $f$
- $y=0$ is a horizontal asymptote of $e^x$.

Theme: $\xrightarrow{\infty}$, $\xrightarrow{-\infty}$

$$\lim_{x \to \infty} \sin\left(\tfrac{1}{x}\right) \cdot x^2 + \frac{1}{x-2} = \underbrace{\lim_{x \to \infty} \sin\left(\tfrac{1}{x}\right) \cdot x^2}_{:= L_1} + \underbrace{\lim_{x \to \infty} \frac{1}{x-2}}_{:= L_2} = L_\infty$$

Question: which one dominates?




$L_1 = \lim_{x \to \infty} \sin\left(\frac{1}{x}\right) \cdot x^2$. Let $u = \frac{1}{x} \Rightarrow \begin{cases} x \to \infty,\ u = \frac{1}{x} \to 0^+ \\ x = \frac{1}{u} \Rightarrow x^2 = \frac{1}{u^2} \end{cases}$

$$= \lim_{u \to 0^+} \left( \sin(u) \cdot \frac{1}{u^2} = \underbrace{\frac{\sin(u)}{u}}_{=1} \cdot \underbrace{\frac{1}{u}}_{\infty} \right) = +\infty$$

$L_2 = \lim_{x \to \infty} \frac{1}{x-2} = 0.$ ✓

$\lim_{x \to -\infty} \sin\left(\frac{1}{x}\right) \cdot x^2 + \frac{1}{x-2}$.

$L_\infty = +\infty + 0 = +\infty$

$L_{-\infty} = -\infty + 0 = -\infty$

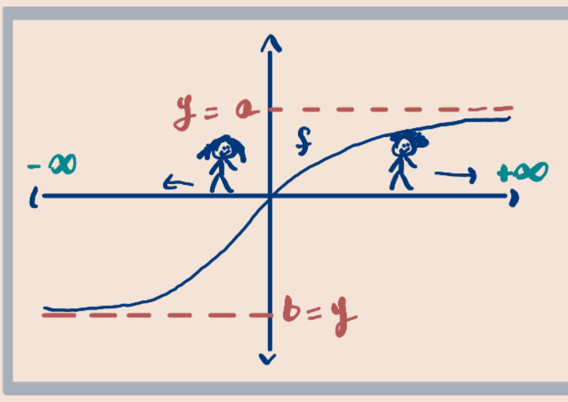
2. Which of the following is a horizontal asymptote of $y = \frac{4x}{\log(|x|^7) + 7x}$?

a) $y = \frac{4}{7}$ b) $y = \frac{4}{7^2}$ c) $y = 0$ d) No Asymptote

$\boxed{\lim_{x \to \pm\infty} y}$

$$y = \frac{4x}{\log(|x|^7) + 7x} = \frac{4x}{7\log|x| + 7x} = \frac{4}{7} \cdot \frac{x}{\log|x| + x}$$

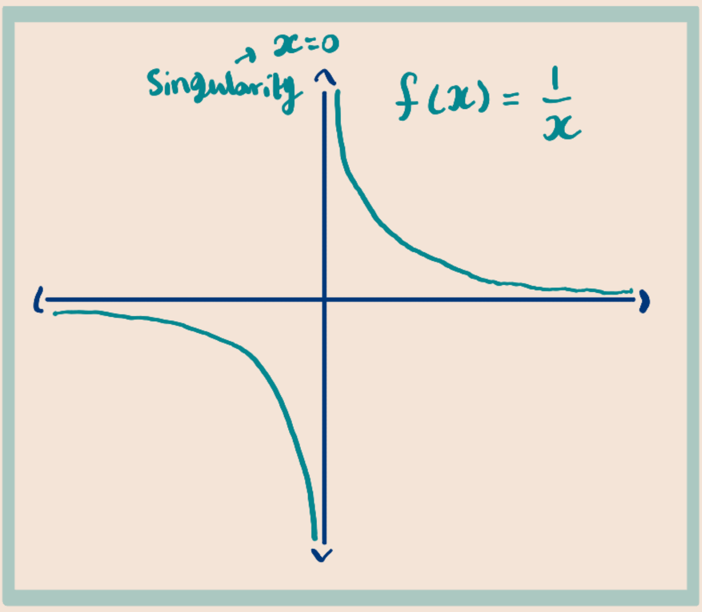
$$L = \frac{4}{7} \cdot \left( \lim_{x \to \pm\infty} \frac{x}{\log|x| + x} \overset{(*)}{=} \lim_{x \to \pm\infty} \frac{\cancel{x}}{\cancel{x}} = 1. \right) = \boxed{\frac{4}{7}}.$$




$(*)$ $\lim_{x \to \pm\infty} \frac{1}{\underbrace{\frac{\log|x|}{x}}_{0} + 1}$

↳ Reason: $\log(x)$ grows slower than any polynomial!

3. Find all vertical asymptotes of $y(x) = \sin(\underset{\substack{\uparrow \\ x=0}}{\frac{1}{x}}) \cdot x^2 + \underset{\substack{\uparrow \\ x=2}}{\frac{1}{x-2}}$




Singularity Check:

- $\frac{1}{x}$ in $\sin(\frac{1}{x})$ $(x=0) \Rightarrow L_0 = \lim_{x \to 0} \sin(\frac{1}{x}) \cdot x^2 + \frac{1}{x-2}$
- $\frac{1}{x-2}$ in $\frac{1}{x-2}$ $(x=2) \Rightarrow L_2 = \lim_{x \to 2} \sin(\frac{1}{x}) \cdot x^2 + \frac{1}{x-2}$

$\boxed{(x=0) \text{ is not a vertical asymptote!}}$

$L_2 = \lim_{x \to 2} \sin(\frac{1}{x}) \cdot x^2 + \lim_{x \to 2} \frac{1}{x-2}$ $\boxed{u = x-2}$

$= \sin(\frac{1}{2}) \cdot 4 + \lim_{u \to 0} \frac{1}{u}$

$\Rightarrow$ $(x=2)$ is a vertical asymptote

$L_0 = \underbrace{\lim_{x \to 0} \sin(\frac{1}{x}) \cdot x^2}_{=L} + \underbrace{\lim_{x \to 0} \frac{1}{x-2}}_{} = -\frac{1}{2}$ $\Uparrow$

$L_0 = L - \frac{1}{2}$

$L = \lim_{x \to 0} \sin(\frac{1}{x}) \cdot x^2$ $\boxed{u = \frac{1}{x}}$

$= \lim_{u \to \pm\infty} \frac{\overbrace{\sin(u)}^{[-1,1]}}{u^2} = 0$

4. An oblique asymptote $y = ax + b$ may be found by finding coefficients from the asymptotic behaviour $\begin{cases} a = \lim_{x \to \pm\infty} \frac{y(x)}{x} \\ b = \lim_{x \to \pm\infty} y(x) - ax \end{cases}$. Find the oblique asymptote of

$y(x) = \sin(\frac{1}{x}) \cdot x^2 + \frac{1}{x-2}$.

$\boxed{y = x}$

Hint: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$a = \lim_{x \to \pm\infty} \left( \frac{y(x)}{x} = \left(\sin(\frac{1}{x}) \cdot x^2 + \frac{1}{x-2}\right) \frac{1}{x} = \sin(\frac{1}{x}) \cdot x + \frac{1}{x} \cdot \frac{1}{x-2} \right)$

$= \lim_{x \to \pm\infty} \sin(\frac{1}{x}) \cdot x \underset{u = \frac{1}{x}}{=} \lim_{u \to 0} \sin u \cdot \frac{1}{u} + \lim_{x \to \pm\infty} \frac{1}{x} \cdot \frac{1}{x-2} = 1 + 0 = \boxed{1 = a}$

$\lim_{x \to \pm\infty} \left( y(x) - ax = y(x) - x = \sin(\frac{1}{x}) \cdot x^2 + \frac{1}{x-2} - x \right) \underset{u = \frac{1}{x}}{=} \lim_{u \to 0} \sin(u) \cdot \frac{1}{u^2} - \frac{1}{u}$

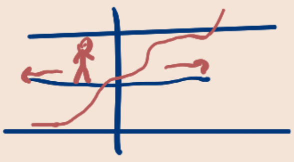
$= \lim_{u \to 0} \frac{1}{u} \left( \underbrace{\sin(u) \cdot \frac{1}{u}}_{\to 1} - 1 \right) = \lim_{u \to 0} \frac{1}{u} \cdot \underbrace{(1 - 1)}_{=0} = \boxed{0 = b}$

5. Find all horizontal asymptotes of

$$y(x) = \frac{1}{x}\cdot\left[\ln\left(\frac{1}{x}\right) + \ln(x)\right)\cdot\left(x^2+\underbrace{x+2x}_{3x}\right)+x+\ln(x)\Big]$$

$$\boxed{\ln(a)+\ln(b) = \ln(a\cdot b)}$$

$$\boxed{\ln(1) = 0}$$



$$y(x) = \frac{1}{x}\cdot\left[\left(\ln\left(\frac{1}{x}\right)+\ln(x)\right)\cdot\left(x^2+3x\right)+x+\ln(x)\right]$$

$$= \cancel{\frac{1}{x}}\cdot\left(\ln\left(\frac{1}{x}\right)+\ln(x)\right)\cdot\left(x^{\cancel{2}}+3\cancel{x}\right)+\cancel{x}\cdot\frac{1}{x}+\ln(x)\cdot\frac{1}{x}$$

$$= \left(\ln\left(\frac{1}{x}\right)+\ln(x)\right)\cdot(x+3)+1+\ln(x)\cdot\frac{1}{x}$$

$$= \cancel{\underbrace{\ln\left(\frac{1}{\cancel{x}}\cdot\cancel{x}\right)}_{=0}\cdot\overset{=0}{(x+3)}}+1+\ln(x)\cdot\frac{1}{x}$$

$$\lim_{x\to\pm\infty}\left(y(x) = 1+\frac{\ln(x)}{x}\right) = 1+\lim_{x\to\pm\infty}\frac{\ln(x)}{x} = 1+0 = \boxed{1}$$

6. Evaluate the limit $\lim\limits_{y\to 2}\dfrac{2-y}{\frac{1}{2}-\frac{1}{y}}$.

$$\lim_{y\to 2}\left(\frac{2-y}{\frac{1}{2}-\frac{1}{y}}\cdot\overset{=1}{\frac{2y}{2y}} = \frac{2y\overset{(-1)}{\cancel{(2-y)}}}{\cancel{y-2}} = -2y\right) = -2\cdot 2 = \boxed{-4}$$

$$(2-y) = -(y-2)$$

7. Evaluate the limit $\lim_{y\to0} \frac{\sqrt{2+y}-\sqrt{2-y}}{4y}$.

$\boxed{a^2-b^2=(a-b)(a+b)}$

$$\lim_{y\to0}\left(\frac{\overset{a}{\sqrt{2+y}}-\overset{b}{\sqrt{2-y}}}{4y}\cdot\overbrace{\frac{\sqrt{2+y}+\sqrt{2-y}}{\sqrt{2+y}+\sqrt{2-y}}}^{=1}\right)$$

$$\left(=\frac{\cancel{2}+y-\cancel{2}+y}{(\sqrt{2+y}+\sqrt{2-y})\cdot 4y}=\frac{\cancel{2y}}{2\cancel{4y}(\sqrt{2+y}+\sqrt{2-y})}\right)$$

$$=\lim_{y\to0}\frac{1}{2}\cdot\frac{1}{\sqrt{2+y}+\sqrt{2-y}}$$

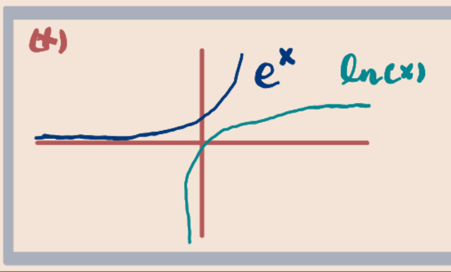
$$\boxed{=\frac{1}{4}\cdot\frac{1}{\sqrt{2}}=\frac{1}{4}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{8}.}$$

8. Evaluate the limit $\lim_{x\to0}\sqrt{x}\ln(x)+e^x x^3=\underbrace{\lim_{x\to0}\sqrt{x}\ln(x)}_{:=L_1}+\underbrace{\lim_{x\to0}e^x x^3}_{:=L_2}=0.$

$$L_2=\lim_{x\to0}e^x x^3=e^0\cdot0^3=0$$

$$L_1=\lim_{x\to0}\left(\sqrt{x}\,\underbrace{\ln(x)}=x^{\frac{1}{2}}\cdot\ln(x)\right)\overset{(*)}{=}0$$

(*) $\underbrace{\ln(0)}=a \Leftrightarrow \underbrace{e^a}=0$




(*) ln(x) grows slower than any polynomial.

$\lim_{x\to0}x^{\frac{1}{2}}\cdot\ln(x)$, $u=x^{\frac{1}{2}}\xrightarrow{x\to0}0$

$\lim_{u\to0}\left(\underline{u}\cdot\ln(u^2)=2\,u\cdot\ln(u)\right)=0$, $w=\frac{1}{u}\xrightarrow{u\to0}\pm\infty$

$\lim_{w\to\pm\infty}\left(\frac{1}{w}\cdot\ln\left(\frac{1}{w^2}=w^{-2}\right)=-2\cdot\frac{\ln(w)}{w}\right)=-2\cdot0=0$

9. Evaluate the limit $\lim_{x\to0}\left(\frac{1-\cos x}{x\tan x}=y\right)$

$$y=\frac{1-\cos x}{2}\cdot\frac{2}{x\tan x}=\sin^2\left(\frac{x}{2}\right)\cdot 2\cdot\frac{1}{x}\cdot\frac{\cos x}{\sin x}$$

$\rightarrow$ Hint: $\lim_{x\to0}\frac{\sin x}{x}=1$

$$\boxed{\frac{1-\cos 2\theta}{2}=\sin^2\theta}$$

$$=\underbrace{\frac{\sin(\frac{x}{2})}{\frac{x}{2}}}_{\to1}\cdot\underbrace{\frac{\sin(\frac{x}{2})}{\frac{x}{2}}}_{\to1}\cdot\underbrace{\frac{1}{4}\cdot\frac{x}{\sin x}}_{\to1}\cdot\underbrace{\cos x}_{\to1}\cdot 2$$

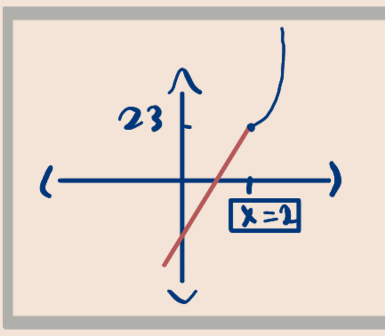
$$\frac{2}{4}\lim_{x\to0}\frac{\sin(\frac{x}{2})}{\frac{x}{2}}\cdot\lim_{x\to0}\frac{\sin(\frac{x}{2})}{\frac{x}{2}}\cdot\lim_{x\to0}\frac{x}{\sin x}\cdot\lim_{x\to0}\cos x$$

$$=\frac{1}{2}(1)(1)\left(\frac{1}{\left(\lim_{x\to0}\frac{\sin x}{x}=1\right)}=1\right)(\underbrace{\cos 0}_{=1})=\boxed{\frac{1}{2}}$$

10. Let $f(x)=\begin{cases}kx+7 & \boxed{x\ge2}\\ x^2+19 & \boxed{x<2}\end{cases}$. For what $k$ is $\lim_{x\to2}f(x)$ defined?

$\lim_{x\to2}f(x)$ defined $\Rightarrow \lim_{x\to2}f(x)=f(2)=kx+7$

$\left(\lim_{x\to x_0}f(x)=f(x_0)\right)$




$$\lim_{x\to2^+}f(x)=kx+7=2k+7$$
$$\Rightarrow \boxed{k=8}$$
$$\lim_{x\to2^-}\left(f(x)=x^2+19\right)=2^2+19=23$$