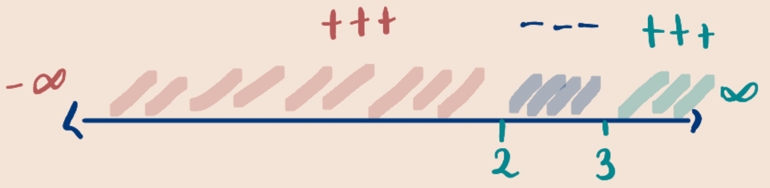
1. For which x is $x^2 - 5x + 6 \geq 0$?

$$x^2 - 5x + 6 = 0 = \underbrace{(x-3)(x-2)}_{f(x)}$$

Inspection:



closed.

$x \in (-\infty, 2]$

$[3, \infty)$ does not include ∞!

open

2. Solve $5x^2 = x^4 - 14$.

$\Rightarrow x^4 - 5x^2 - 14 = 0$ Claim: degree 2 polynomial in x^2!

$\Rightarrow \underbrace{(x^2)^2}_{} - 5(x^2) - 14 = 0$, let $x^2 = u$

$= x^{2 \cdot 2} = x^4$

roots $= \{\sqrt{2}i, -\sqrt{2}i, \sqrt{7}, -\sqrt{7}\}$

$\Rightarrow u^2 - 5u - 14 = 0 = (u-7)(u+2)$

$u = 7$ or $u = -2$

$x^2 = 7$ $\quad$ $x^2 = -2$ if $x = \pm\sqrt{2}\, i$

if $x = \pm\sqrt{7}$

3. Solve $x^4 + 4x^2 = 20x + x^3 - 16$. Hint: $r_1 = 2$ is a solution.

$x^4 - x^3 + 4x^2 - 20x + 16 = 0 = (\cancel{x - r_1})(\cancel{x - r_2})(x - r_3)(x - r_4) = \dfrac{x^3 + x^2 + 6x - 8}{(x-1)}$

$(\cancel{x - r_2})$ $\quad (x - r_1) \rightarrow$ Polynomial long division $\quad (\cancel{x - r_1})\ (\cancel{x - r_2})$

$= x^2 + 2x + 8 = 0$

$$
\begin{array}{r l}
 & x^3 + x^2 + 6x - 8 \\
x - 2 \,\big) & \cancel{x^4} - x^3 + 4x^2 - 20x + 16 \\
 & -(\cancel{x^4} - 2x^3) \\
 & \cancel{x^3} + 4x^2 - 20x + 16 \\
 & -(\cancel{x^3} - 2x^2) \\
 & \cancel{6x^2} - 20x + 16 \\
 & -(\cancel{6x^2} - 12x) \\
 & \cancel{-8x + 16} \\
 & -(\cancel{-8x + 16})
\end{array}
$$

Guessing! $r_2 = 1$

$$
\begin{array}{r l}
 & x^2 + 2x + 8 \\
x - 1 \,\big) & x^3 + x^2 + 6x - 8
\end{array}
$$

$\Rightarrow x = -1 \pm i\sqrt{7}$

$\begin{cases} a = 1 \\ b = 2 \\ c = 8 \end{cases}$

$\Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

1, 2,

4. Which function has the roots $x_{1,2,3,4} = \{-4, 5, 7, 1\}$?

(a) ~~$f(x) = x^4 + 7x^3 - 21x^2 - 167x - 140$~~

(b) ~~$f(x) = x^4 + 17x^3 + 99x^2 + 223x + 140$~~

(c) $f(x) = x^4 - 9x^3 - 5x^2 + 153x - 140$

(d) ~~$f(x) = x^4 + 5x^3 - 33x^2 - 113x + 140$~~

$f(1) = \underbrace{1 + 7} - 21 - 167 - 140 \neq 0$

$f(1) = 1 + 17 + 99 + 223 + 140 \neq 0$

$\boxed{f(7) = x^4 - 9x^3 - 5x^2 + 153x - 140}$

$f(7) = 7^4 + 5 \cdot 7^3 - 33 \cdot 7^2 - 113 \cdot 7 + 140 > 0$

---

5. Given $f(x) = 1x^3 + bx^2 + cx \boxed{-14}$ with $b, c \in \mathbb{R}$, and $f(1+i) = 0$, solve $f(x) = 0$.

$\mathbb{R}$ $\mathbb{R}$ $\mathbb{R}$

1. $f(x)$ is a polynomial with real coefficients
2. $z \in \mathbb{C}$ is a root

$\Rightarrow z^*$ is a root.

$\boxed{z = 1+i}$ is a root $\Rightarrow$ $\boxed{z^* = 1-i}$ is a root.

Observation: $n=3$, so 3 roots, by FTA

$f(x) = (x-(1+i))(x-(1-i))(x-r_3) \Rightarrow 14 = (1+i)(1-i) r_3$

$14 = 2 \cdot r_3$

$\boxed{\Rightarrow r_3 = 7}$

---

6. Find all the roots (real or complex) of the Polynomial

$P(x) = x^6 - 4x^5 - x^4 + 18x^3 - 18x^2 - 8x + 24 = (x-r_1)(x-r_2)(x-r_3)(x-r_4)(x-r_5)(x-r_6)$

~~a) $x_{1,2,3,4,5,6} = \{-2, -1, 1, 3, 1+i, 1-i\}$~~

~~b) $x_{1,2,3,4,5,6} = \{2, 1, 1, 3, 2+i, 2-i\}$~~ $P(1) \neq 0 \Rightarrow 1$ is not a root!

c) $x_{1,2,3,4,5,6} = \{-2, -1, 2, 3, 1+i, 1-i\}$ $\Rightarrow \begin{cases} P(-2) = 0 \\ \cancel{P(-3) = 0} \end{cases}$ $\boxed{C}$

~~d) $x_{1,2,3,4,5,6} = \{-3, -1, 2, 3, 1+i, 1-i\}$~~

-3 a root $\Rightarrow$ $\dfrac{P(x)}{x-(-3)}$ yields no remainder!

$x^5 - 7x^4 + 20x^3 - 42x^2 + 108x - 332$ rem $= 1020 \neq 0 \Rightarrow -3$ is not a root!

$x+3 \mid \cancel{x^6} - 4x^5 - x^4 + 18x^3 - 18x^2 - 8x + 24$.

$-(\cancel{x^6} + 3x^5)$

$-7\cancel{x^5}$

$-(-7\cancel{x^5} - 21x^4)$

$20x^4$

$\cancel{20x^4}$

$-(\cancel{20x^4} + 60x^3)$

$-42\cancel{x^3}$

$-(-42\cancel{x^3} - 126x^2)$

$108x^2$

$\cancel{108x^2}$

$-(\cancel{108x^2} + 324x)$

$-332x$

$-(-332x - 996)$

rem $= 1020$

Ex: Check that $\dfrac{P(x)}{x+2}$ has no remainder!

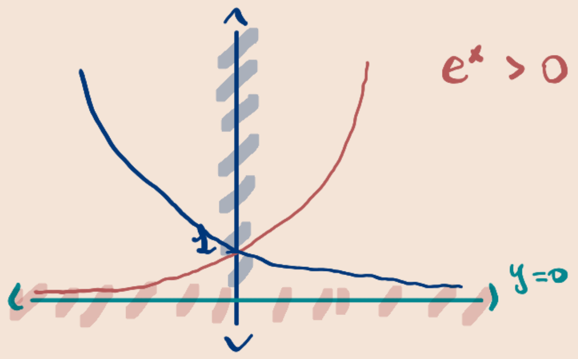
$x^5 - x^4 + 2x^3 + 12x^2 - 54x + 154$

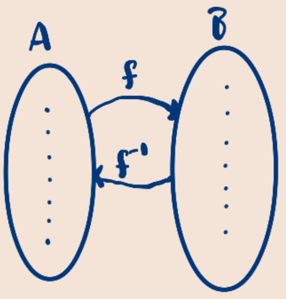
7. Let $f(x) = e^{-9x+3}$. Determine the Domain & Range of f, f^{-1}.

Question: What x can we not input? None.

$$\begin{cases} D_f = (-\infty, \infty) \\ R_f = (0, \infty) \end{cases}$$

Fact: $e \simeq 2.71$ is positive.





Domain = "all possible input"
Range = "all possible output"

$D_f = A = R_{f^{-1}}$

$R_f = B = D_{f^{-1}}$

8. Let $g(x) = e^{-13x^2+7}$. Determine the Domain & Range of g.

$\boxed{D_g = (-\infty, \infty) \ ; \ R_g = (0, e^7]}$ ↳ closed

$$\begin{cases} \text{Domain = "all possible input"} \\ \text{Range = "all possible output"} \end{cases}$$

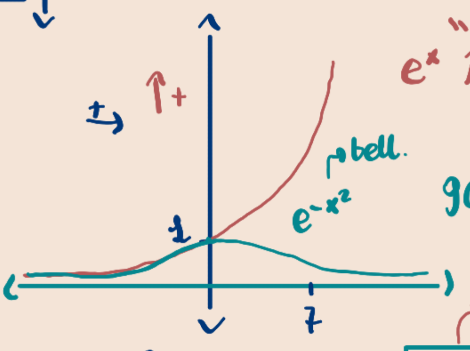
$g(x) = e^{-13x^2+7} = e^7 \cdot (e^{-x^2})^{13}$

Behaviour of e^{-x^2}: $e^{\boxed{-x^2}} \downarrow$

if $x \uparrow$ then $\boxed{-x^2} \downarrow$

≥ 0

≤ 0



e^x "↗": $y > x \Rightarrow e^y > e^x$.

$g(x=0) = e^7$

‖

maxima

which x gives $\max(e^{-x^2})$?

The same x which gives $\max(-x^2)$ which is $\boxed{x=0.}$

9. Consider $(y-2)^2 = 2(x-1)$. Which of the following is true?

a) y is not a function of x, x is not a function of y

b) y is not a function of x, x is a function of y

c) y is a function of x, x is not a function of y

d) y is a function of x, x is a function of y

Function: takes an input and returns an output!

$(y-2)^2 = 2(x-1) \Rightarrow y-2 = \pm\sqrt{2(x-1)} \Rightarrow y = 2 \pm \sqrt{2(x-1)}$

for $x \neq 1$ we have 2 outputs $\Rightarrow$ y cannot be a function of x.

$(y-2)^2 = 2(x-1) \Rightarrow (y-2)^2 \cdot \frac{1}{2} + 1 = x = f(y)$ $\begin{cases} \text{input } y \\ \text{output } x\text{, only one!} \end{cases}$

$= x$

10. Find the value of $\sum_{k=0}^{n} \binom{n}{k}$.

Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k y^{n-k}$

$x = y = 1$. Then

$(1+1)^n = (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot \underbrace{1^k}_{=1} \underbrace{1^{n-k}}_{=1} = \sum_{k=0}^{n} \binom{n}{k}$

$\boxed{= 2^n.}$

$n=0$ | $1_{k=0}$

$n=1$ | $1_{k=0}$ $1_{k=1}$

$n=2$ | 1_0 2_1 1_2

$n=3$ | 1_0 3_1 3_2 1_3

$\vdots$ $\vdots$

EX

$\binom{3}{1}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$(x+y)^3 = 1x^3 + 3x^2y + 3yx^2 + 1y^3$