1. Find the inverse of $A = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} \boxed{2} & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right] \xrightarrow{\times \frac{1}{2}} = \left[\begin{array}{ccc|ccc} \boxed{1} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right] \times -1$$

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \boxed{-\frac{1}{4}} & \frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right] \to \times -4$$

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \boxed{1} & -2 & 0 & -4 & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \times -\frac{1}{4} \\ \times -\frac{1}{2} \end{array}$$

Draft.

row:	0	0	$\boxed{1}$	|	2	4	-2	
$\times -1$ new row:	0	0	-1	|	-2	-4	2	
+ other row:	$\boxed{1}$	0	1	|	0	1	0	
	1	0	0	|	-2	-3	2	

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & -2 & 0 & -4 & 2 \\ 0 & 0 & \boxed{\frac{1}{2}} & 1 & 2 & -1 \end{array}\right] \xrightarrow{\times 2} = \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & -2 & 0 & -4 & 2 \\ 0 & 0 & \boxed{1} & 2 & 4 & -2 \end{array}\right] \begin{array}{l} \times -1 \\ \times 2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & -2 & -3 & 2 \\ 0 & \boxed{1} & 0 & 4 & 4 & -2 \\ \underbrace{0 \quad 0}_{:= Id} & & \boxed{1} & 2 & 4 & -2 \end{array}\right] \Rightarrow A^{-1} = \begin{pmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{pmatrix} \Rightarrow A \cdot A^{-1} = Id$$

$$\begin{pmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{pmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} [-2\ -3\ 2]\begin{bmatrix}0\\1\\2\end{bmatrix} & [-2\ -3\ 2]\begin{bmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{bmatrix} & [-2\ -3\ 2]\begin{bmatrix}-\frac{1}{2}\\1\\1\end{bmatrix} \\ [4\ 4\ -2]\begin{bmatrix}0\\1\\2\end{bmatrix} & [4\ 4\ -2]\begin{bmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{bmatrix} & [4\ 4\ -2]\begin{bmatrix}-\frac{1}{2}\\1\\1\end{bmatrix} \\ [2\ 4\ -2]\begin{bmatrix}0\\1\\2\end{bmatrix} & [2\ 4\ -2]\begin{bmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{bmatrix} & [2\ 4\ -2]\begin{bmatrix}-\frac{1}{2}\\1\\1\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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2. Find the inverse of $A = \begin{bmatrix} 3.5 & -1 & 0.5 \\ 10 & -3 & 2 \\ 2.5 & -1 & 1.5 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & -1 & \frac{1}{2} \\ 10 & -3 & 2 \\ \frac{5}{2} & -1 & \frac{3}{2} \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} \boxed{\frac{7}{2}} & -1 & \frac{1}{2} & 1 & 0 & 0 \\ 10 & \boxed{-3} & 2 & 0 & 1 & 0 \\ \frac{5}{2} & -1 & \boxed{\frac{3}{2}} & 0 & 0 & 1 \end{array}\right] \xrightarrow{\times \frac{2}{7}} = \left[\begin{array}{ccc|ccc} \boxed{1} & -\frac{2}{7} & \frac{1}{7} & \frac{2}{7} & 0 & 0 \\ 10 & -3 & 2 & 0 & 1 & 0 \\ \frac{5}{2} & -1 & \frac{3}{2} & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \times -10 \\ \times -\frac{5}{2} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & -\frac{2}{7} & \frac{1}{7} & \frac{2}{7} & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{4}{7} & -\frac{20}{7} & 1 & 0 \\ 0 & -\frac{2}{7} & \frac{8}{7} & -\frac{5}{7} & 0 & 1 \end{array}\right] \times -2$$

Draft.

row:	0	$-\frac{1}{7}$	$\frac{4}{7}$	|	$-\frac{20}{7}$	1	0	
$\times -2$ new row:	0	$+\frac{2}{7}$	$-\frac{8}{7}$	|	$+\frac{40}{7}$	-2	0	
+ other row:	0	$-\frac{2}{7}$	$\frac{8}{7}$	|	$-\frac{5}{7}$	0	1	
	0	0	0	|	5	-2	1	

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & -\frac{2}{7} & \frac{1}{7} & \frac{2}{7} & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{4}{7} & -\frac{20}{7} & 1 & 0 \\ \boxed{0 \quad 0 \quad 0} & & & -\frac{5}{7} & 0 & 1 \end{array}\right]$$

$\Rightarrow$ A does not have an inverse!

Determinant test

$\det(A) = 0 \iff A$ is not invertible

3. Find the kernel of $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

$\operatorname{Ker}(A) = \{ \text{all } \vec{x} : A\vec{x} = \vec{0} \}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Set $A\vec{x} = 0$, solve!

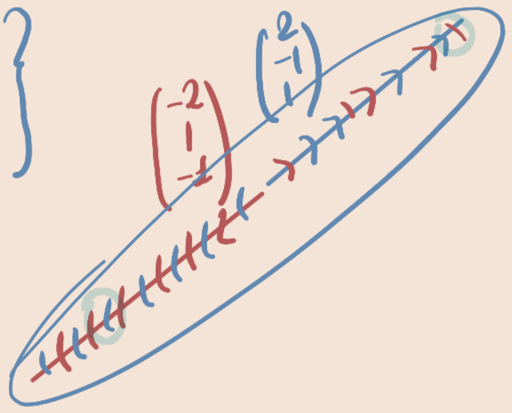
$$\iff \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \iff \left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \end{array}\right] \Big\} \times -2 \iff \left[\begin{array}{ccc|c} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 1 & 0 \end{array}\right]$$

$0x_1 + 0x_2 + 0x_3 = 0$

$$\iff \left[\begin{array}{cc|cc} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & -1 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = 2x_3 \\ x_2 = -x_3 \end{array} \iff \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = \boxed{x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}$$

$$\operatorname{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$-1 \times \mathbb{R} = \mathbb{R}$



4. Find the inverse of $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = A$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \times 1 \\ \times -1 \end{matrix} = \left[\begin{array}{ccc|ccc} \boxed{1} & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & \boxed{2} & 0 & -1 & 0 & 1 \end{array}\right] \times \tfrac{1}{2}$$

Draft.

	row:	0	0	2	1	1	0
$\times -\frac{1}{2}$	new row:	0	0	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0
+	other row:	1	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$
		1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \boxed{2} & 1 & 1 & 0 \\ 0 & \boxed{2} & 0 & -1 & 0 & 1 \end{array}\right] \times -\tfrac{1}{2}$$

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \boxed{2} & 1 & 1 & 0 \\ 0 & \boxed{2} & 0 & -1 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \boxed{2} & 1 & 1 & 0 \\ 0 & \boxed{2} & 0 & -1 & 0 & 1 \end{array}\right] \begin{matrix} \to \times \frac{1}{2} \\ \to \times \frac{1}{2} \end{matrix}$$

$$= \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \boxed{1} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \boxed{1} & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}$$

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5. Given the matrix $A = B \cdot C \cdot D$, find A^{-1} in terms of B, C, D.

$$A^{-1} = (B_{m\times n} \cdot C_{n\times q} \cdot D_{q\times s})^{-1} = \boxed{D^{-1}_{s\times q} \cdot C^{-1}_{q\times n} \cdot B^{-1}_{n\times m}}$$

$B_{m\times n} \Rightarrow B^{-1}_{n\times m}$

$C_{n\times q} \Rightarrow C^{-1}_{q\times n}$

$D_{q\times s} \Rightarrow D^{-1}_{s\times q}$

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6. An $n\times k$ matrix has

$$Ker(A) = Span\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

What is the dimension of its image? $\boxed{\dim(im(A)) = k - 2}$

- $A_{n\times k}: V \to W$, $\dim(V) = n$, $\dim(W) = k$

- $Ker(A) = \{\text{all } \vec{x} : A\vec{x} = \vec{0}\} = Span\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$= \left\{ \alpha_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \alpha_1, \alpha_2 \in \mathbb{F} \right\}$$

$\dim(Ker(A)) = 2$

7. How is the map $D=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ represented in the basis

$\mathcal{B}=\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix},\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right\}$?

$$D\cdot\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=1\cdot\begin{bmatrix}0\\1\end{bmatrix}+1\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$D\cdot\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=1\cdot\begin{bmatrix}0\\1\end{bmatrix}+-1\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\boxed{D_{new}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}}$$

$$D=\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\},\quad id=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

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8. Which matrix represents a counter-clockwise rotation around the z-axis?

Note: $\mathcal{B}=\{e_x,e_y,e_z\}$

A. $\cancel{\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\\sin\theta&0&\cos\theta\end{bmatrix}}$

B. $\cancel{\begin{bmatrix}1&0&-\sin\theta\\\cos\theta&1&\cos\theta\\\sin\theta&0&1\end{bmatrix}}$

C. $\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$

D. $\cancel{\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&0\end{bmatrix}}$

$$\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}\begin{pmatrix}e_x\\e_y\\e_z\end{pmatrix}=e_x\begin{pmatrix}\cos\theta\\\sin\theta\\0\end{pmatrix}+e_y\begin{pmatrix}-\sin\theta\\\cos\theta\\0\end{pmatrix}+\begin{pmatrix}0\\0\\e_z\end{pmatrix}$$

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9. Consider $H_2(\mathbb{C})=\{A\in\mathbb{C}^{2\times2}: A^\dagger=A\}$. What is a suitable choice of field $\mathbb{F}$? $(\mathbb{R},\mathbb{C})$

Vector space checklist.

0. Addition and scalar multiplication are commutative, associative. ☑
1. V is closed under addition and scalar multiplication. ☐
2. There exists a zero vector, neutral element of addition in V ☑
3. Scaling by $1\in\mathbb{F}$ yields the same vector ☑

$r\cdot A\in H_2(\mathbb{C})$? $\boxed{\mathbb{R}=\mathbb{F}}$

$(r\cdot A)^\dagger=r\cdot A$?

$A=\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}=\begin{pmatrix}\bar{a}_1&\bar{a}_3\\\bar{a}_2&\bar{a}_4\end{pmatrix}=:A^\dagger$

$r\cdot A^\dagger=\begin{pmatrix}r\cdot\bar{a}_1&r\cdot\bar{a}_3\\r\cdot\bar{a}_2&r\cdot\bar{a}_4\end{pmatrix}=r\cdot A$ ✓

$c\cdot A\in H_2(\mathbb{C})$? $\cancel{\mathbb{F}=\mathbb{C}}$

$(c\cdot A)^\dagger=\bar{c}\cdot A$

$\bar{c}\cdot A^\dagger=\begin{pmatrix}\overline{a_1\cdot c}&\overline{a_3\cdot c}\\\overline{a_2\cdot c}&\overline{a_4\cdot c}\end{pmatrix}\notin H_2(\mathbb{C})$

$\parallel$

$\neq\boxed{c\cdot A^\dagger}$

10. Consider $P_2(\mathbb{R}) = \{p(x) : p(x) \text{ is a quadratic polynomial}\}$. Is the derivative operator $\mathcal{D}: p(x) \mapsto p'(x) := \frac{d}{dx} p(x)$ a linear operator?

$$B = \{1, x, x^2\} \quad a \cdot x^2 + b \cdot x + c \cdot 1 = p(x) = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$$

$$\mathcal{D}(p(x)) = 0 \cdot x^2 + 2ax + b \cdot 1 = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$$

1) $\mathcal{D}(\lambda p(x)) = \lambda \cdot \mathcal{D}(p(x))$ ✓ $\boxed{\text{YES}}$

2) $\mathcal{D}(p(x) + g(x)) = \mathcal{D}(p(x)) + \mathcal{D}(g(x))$ ✓

$\mathcal{D}$ is the matrix where

$$\begin{bmatrix} & \mathcal{D} & \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$$

$$B = \{1, x, x^2\}, \; 1 \cdot 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; 1 \cdot x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; 1 \cdot x^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$B = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{:= I_d}$$

$$\mathcal{D}(1) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{D}(x) = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array} := \mathcal{D}$$

$$\mathcal{D}(x^2) = 2 \cdot x = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} \checkmark = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c \\ b \\ a \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right)$$

$$= \cancel{c \cdot \overset{=0}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2a \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix} !$$