1. $\begin{bmatrix} 1 & 2 & 9 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{pmatrix} -6 & 10 & 8 \\ 2 & 6 & 4 \\ 5 & 9 & 7 \end{pmatrix}$

$$\begin{pmatrix} (1\,2\,9)\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix} & (1\,2\,9)\begin{pmatrix}-1\\1\\1\end{pmatrix} & (1\,2\,9)\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix} \\ (3\,4\,5)\begin{pmatrix}1\\1\\-1\end{pmatrix} & (3\,4\,5)\begin{pmatrix}-1\\1\\1\end{pmatrix} & (3\,4\,5)\begin{pmatrix}1\\-1\\1\end{pmatrix} \\ (6\,7\,8)\begin{pmatrix}1\\1\\-1\end{pmatrix} & (6\,7\,8)\begin{pmatrix}-1\\1\\1\end{pmatrix} & (6\,7\,8)\begin{pmatrix}1\\-1\\1\end{pmatrix} \end{pmatrix} = \begin{pmatrix} -6 & 10 & 8 \\ 2 & 6 & 4 \\ 5 & 9 & 7 \end{pmatrix}$$

Example.

$$\boxed{(1\,2\,9)\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix} = 1\cdot 1 + 2\cdot 1 + 9\cdot -1 \\ = -6}$$

Why this works...

$$x\begin{pmatrix}1\\1\\-1\end{pmatrix} + y\begin{pmatrix}-1\\1\\1\end{pmatrix} + z\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\left(\begin{pmatrix}1\\1\\-1\end{pmatrix}\begin{pmatrix}-1\\1\\1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right) = \begin{pmatrix} x\cdot 1 + y\cdot -1 + z\cdot 1 \\ x\cdot 1 + y\cdot 1 + z\cdot -1 \\ x\cdot -1 + y\cdot 1 + z\cdot 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 9 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}\left[\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = \begin{bmatrix} 1 & 2 & 9 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}\left[\begin{pmatrix} \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[1\ -1\ 1] \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[1\ 1\ -1] \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[-1\ 1\ 1] \end{pmatrix}\right]$$

$$= \begin{pmatrix} \begin{pmatrix} \boxed{\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[1\ -1\ 1]} \\ \boxed{\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[1\ 1\ -1]} \\ \boxed{\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[-1\ 1\ 1]} \end{pmatrix}\cdot[1\ 2\ 9] \\ \begin{pmatrix} \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[1\ -1\ 1] \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[1\ 1\ -1] \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[-1\ 1\ 1] \end{pmatrix}\cdot[3\ 4\ 5] \\ \begin{pmatrix} \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[1\ -1\ 1] \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[1\ 1\ -1] \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[-1\ 1\ 1] \end{pmatrix}\cdot[6\ 7\ 8] \end{pmatrix} = \begin{pmatrix} \begin{matrix} \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[1\ -1\ 1] \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[2\ 2\ -2]^{+} \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[-9\ 9\ 9]^{+} \end{matrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[-6\ 10\ 8] \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[2\ 6\ 4] \\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot[5\ 9\ 7] \end{pmatrix}$$

$$[-6\ 10\ 8] = \left(\begin{pmatrix}1\\1\\-1\end{pmatrix}\begin{pmatrix}-1\\1\\1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)\cdot[1\ 2\ 9]\ ,$$

$$\begin{pmatrix} \begin{pmatrix}1\\1\\-1\end{pmatrix}[1\ 2\ 9] & \begin{pmatrix}-1\\1\\1\end{pmatrix}[1\ 2\ 9] & \begin{pmatrix}1\\-1\\1\end{pmatrix}[1\ 2\ 9] \\ \begin{pmatrix}1\\1\\-1\end{pmatrix}[3\,4\,5] & \begin{pmatrix}-1\\1\\1\end{pmatrix}[3\,4\,5] & \begin{pmatrix}1\\-1\\1\end{pmatrix}[3\,4\,5] \\ \begin{pmatrix}1\\1\\-1\end{pmatrix}[6\,7\,8] & \begin{pmatrix}-1\\1\\1\end{pmatrix}[6\,7\,8] & \begin{pmatrix}1\\-1\\1\end{pmatrix}[6\,7\,8] \end{pmatrix}$$

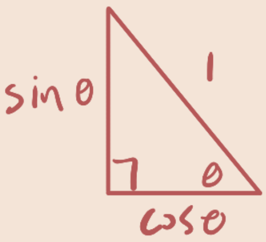
2. Let $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. What is R^{-1}?

$$R \cdot R^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ni A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $\det A = ad - bc$

$$\boxed{R^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix}} \cdot \frac{1}{\underbrace{\det R}_{=1}}$$

$$* \det R = \cos^2\theta + \sin^2\theta = 1$$



$$3. \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \right) \cdot \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \left[[1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \ [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right] \\ \left[[1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \ [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right] \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} \left[[1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \ [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right] \begin{bmatrix} 3 \\ 0 \end{bmatrix} & \left[[1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \ [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right] \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\ \left[[1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \ [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right] \begin{bmatrix} 3 \\ 0 \end{bmatrix} & \left[[1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \ [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right] \begin{bmatrix} 3 \\ 3 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} 3 \cdot [1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \overset{=0}{\cancel{0 \cdot [1\ 1]}} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} & 3 \cdot [1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 3 \cdot [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ 3 \cdot [1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \overset{=0}{\cancel{0 [1\ 1]}} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} & 3 \cdot [1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 3 \cdot [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} 3 \cdot [1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} & 3 \cdot \left([1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} + [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \\ 3 \cdot [1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} & 3 \cdot \left([1\ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} + [1\ 1] \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 18 \\ 12 & 18 \end{bmatrix}$$

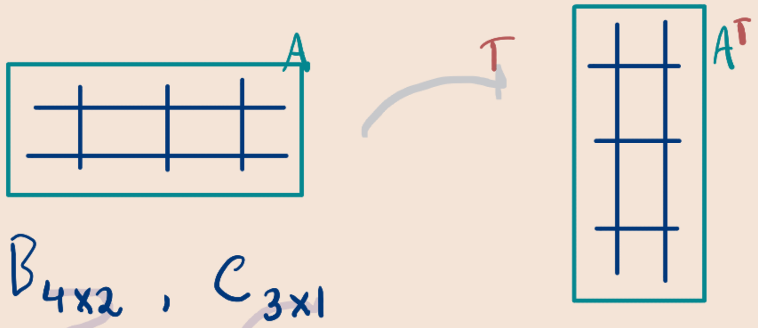
4. $A = \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$, $B = \begin{bmatrix}99&0\\99&99\\99&0\\99&99\end{bmatrix}$, $C = \begin{bmatrix}3\\0\\3\end{bmatrix}$. Which statement is valid?

~~A. A^TB^TC~~

~~B. $C^TB^TA^T$~~

$\boxed{\text{C. } BA^TC}$

~~D. ABC~~



$$A_{3\times2}\ ,\quad B_{4\times2}\ ,\quad C_{3\times1}$$

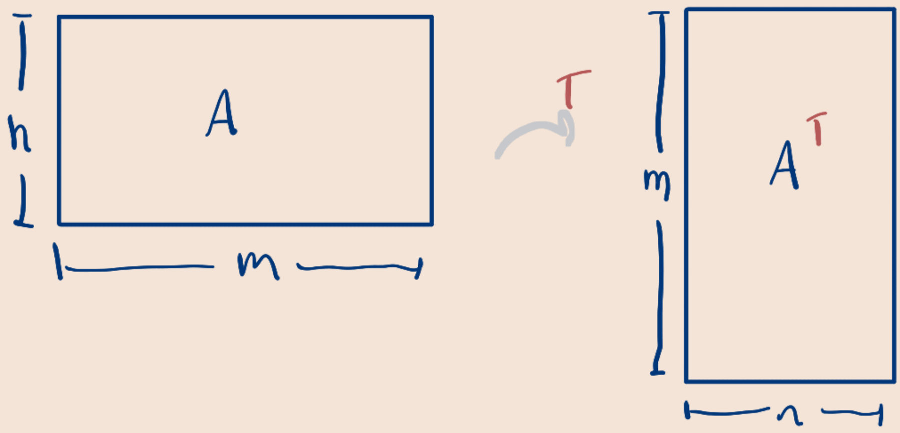
$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$A^T_{2\times3} \qquad B^T_{2\times4} \qquad C^T_{1\times3}$$

---

5. $(A\cdot B\cdot C)^T = ?$

$$A_{m\times n}\cdot B_{n\times p}\cdot C_{p\times q}$$



$A_{m\times n} \Rightarrow A^T_{\cancel{n\times m}}$

$B_{n\times p} \Rightarrow B^T_{\cancel{p\times n}}$

$C_{p\times q} \Rightarrow C^T_{\cancel{q\times p}}$

$$\boxed{(A\cdot B\cdot C)^T = C^T_{q\times p}\, B^T_{p\times n}\, A^T_{n\times m}}$$

---

6. $A_{3\times4}$, A^TB and BA^T are both well-defined. What are the dimensions of B?

$B_{3\times4}$

$A_{3\times4}$: $A^T_{4\times3}\, B_{n\times m} \Rightarrow n=3$

$B_{3\times m}\, A^T_{4\times3} \Rightarrow m=4$

7. Find α such that $\begin{cases} x_1 + \alpha x_2 = 1 \\ x_1 - x_2 + 3x_3 = -1 \\ 2x_1 - 2x_2 + \alpha x_3 = -2 \end{cases}$ has no solutions.

$$\begin{matrix} x_1 + \alpha x_2 = 1 \\ x_1 - x_2 + 3x_3 = -1 \\ 2x_1 - 2x_2 + \alpha x_3 = -2 \end{matrix} \Leftrightarrow \left(\begin{array}{ccc|c} x_1 & x_2 & x_3 & = \\ \boxed{1} & \alpha & 0 & 1 \\ 1 & -1 & 3 & -1 \\ 2 & -2 & \alpha & -2 \end{array}\right) \xrightarrow{\text{goal:}} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & \sim \\ 0 & \boxed{1} & 0 & - \\ 0 & 0 & \boxed{1} & - \end{array}\right) \Leftrightarrow \begin{matrix} x_1 = \sim \\ x_2 = \sim \\ x_3 = \sim \end{matrix}$$

$$\left(\begin{array}{ccc|c} x_1 & x_2 & x_3 & = \\ \boxed{1} & \alpha & 0 & 1 \\ 1 & -1 & 3 & -1 \\ 2 & -2 & \alpha & -2 \end{array}\right) \begin{matrix} \times -1 \\ \\ \times -2 \end{matrix} = \left(\begin{array}{ccc|c} \boxed{1} & \alpha & 0 & 1 \\ 0 & \boxed{-1-\alpha} & 3 & -2 \\ 0 & -2\alpha-2 & \alpha & -4 \end{array}\right) \begin{matrix} \times 1 \\ \\ \times -2 \end{matrix} = \left(\begin{array}{ccc|c} x_1 & x_2 & x_3 & = \\ \boxed{1} & -1 & 3 & -1 \\ 0 & \boxed{-1-\alpha} & 3 & -2 \\ 0 & 0 & \alpha-6 & 0 \end{array}\right)$$

Draft.

$$\begin{array}{rrrrr|r} \times 1 & \text{row:} & 0 & -1-\alpha & 3 & -2 \\ & \text{new row:} & 0 & -1-\alpha & 3 & -2 \\ + & \text{other row:} & 1 & \alpha & 0 & 1 \\ \hline & & 1 & -1 & 3 & -1 \end{array}$$

$$= \left(\begin{array}{ccc|c} x_1 & x_2 & x_3 & = \\ \boxed{1} & -1 & 3 & -1 \\ 0 & \boxed{-1-\alpha} & 3 & -2 \\ 0 & 0 & \alpha-6 & 0 \end{array}\right)$$

$-1-\alpha = 0 \Rightarrow \alpha = -1$

$$\Rightarrow \left(\begin{array}{ccc|c} x_1 & x_2 & x_3 & = \\ \boxed{1} & -1 & 3 & -1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -7 & 0 \end{array}\right) \begin{matrix} \Rightarrow 3x_3 = -2 \Rightarrow x_3 = -\frac{2}{3} \\ \neq \\ \Rightarrow -7x_3 = 0 \Rightarrow x_3 = 0 \end{matrix} \quad ↯$$

---

8. Which is true for Homogeneous Systems of Linear Equations?

$\boxed{\text{A.}}$ $\vec{a}, \vec{b}$ solⁿˢ $\Rightarrow \begin{matrix} A\vec{a} = \vec{0} \\ A\vec{b} = \vec{0} \end{matrix}$. Now consider $A(\vec{a} + \vec{b}) \overset{?}{=} \vec{0} \dots$ $\Leftrightarrow$: $\vec{a} + \vec{b}$ solⁿˢ

$$\underbrace{A\vec{a}}_{:=\vec{0}} + \underbrace{A\vec{b}}_{:=\vec{0}} \overset{✓}{=} \vec{0}$$

Homogeneous
$A\vec{x} = 0$

B. The system might have no solⁿˢ

~~C. $\vec{a}$ solⁿˢ $\Rightarrow A\vec{a} = \vec{0} \Rightarrow A(\vec{a} \cdot k) = k \overbrace{A(\vec{a})}^{=\vec{0}} = \vec{0} \Rightarrow \cancel{\exists}\, k \in \mathbb{R} : k\vec{a}$ not a solⁿˢ~~

D. We can find $\vec{a} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ with all $a_i > 0$

---

9. Let $\vec{a}$ and $\vec{b}$ be solutions to $A\vec{x} = \vec{v}$. When is $\vec{a} + \vec{b}$ a solution?
linear system

Claim: $\vec{v} = \vec{0} \Rightarrow (\vec{a}, \vec{b}$ are solutions $\Rightarrow \vec{a} + \vec{b}$ is a solution$)$

$\vec{0}$ is homogeneous
if $A\vec{a} = 0$

$A\vec{x} = \vec{v} = \vec{0}$

$\vec{a}$ solⁿˢ $\Rightarrow A\vec{a} = \vec{0}$
$\vec{b}$ solⁿˢ $\Rightarrow A\vec{b} = \vec{0}$

$$\boxed{A(\vec{a} + \vec{b}) = \underbrace{A\vec{a}}_{:=\vec{0}} + \underbrace{A\vec{b}}_{:=\vec{0}} = \vec{0}}$$

10. Solve

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 4 \\ x_1 + 4x_2 - 8x_3 = 7 \\ -3x_1 - 7x_2 + 9x_3 = -6 \end{cases} \Leftrightarrow \underbrace{\begin{array}{ccc} x_1 & x_2 & x_3 \\ \multicolumn{3}{c}{\begin{pmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{pmatrix}} \end{array}}_{=A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \overset{=b}{\begin{pmatrix} 4 \\ 7 \\ -6 \end{pmatrix}}$$

① Augment b to A.

$$\left(\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right) \overset{\text{goal:}}{\rightsquigarrow} \left(\begin{array}{ccc|c} \boxed{1} & \textcircled{0} & \textcircled{0} & \sim \\ \textcircled{0} & \boxed{1} & \textcircled{0} & - \\ \textcircled{0} & \textcircled{0} & \boxed{1} & - \end{array}\right)$$

② Gaussian Elimination

$$\left(\begin{array}{ccc|c} \boxed{1} & 3 & -5 & 4 \\ \textcircled{1} & 4 & -8 & 7 \\ \textcircled{-3} & -7 & 9 & 6 \end{array}\right) \begin{matrix} \times -1 \\ \times 3 \end{matrix} = \left(\begin{array}{c|cc|c} \boxed{1} & \textcircled{3} & -5 & 4 \\ 0 & \boxed{1} & -3 & 3 \\ 0 & \textcircled{2} & -6 & 6 \end{array}\right) \begin{matrix} \times -3 \\ \times -2 \end{matrix} = \left(\begin{array}{c|cc|c} \boxed{1} & 0 & 4 & -5 \\ 0 & \boxed{1} & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$0x_1 + 0x_2 + 0x_3 = 0$

Draft.

$$\begin{array}{rl} \times -2 & \begin{array}{l|l|cc|c} \text{row:} & 0 & 1 & -3 & 3 \\ \text{new row:} & 0 & -2 & 6 & -6 \\ + \text{ other row:} & 0 & 2 & -6 & 6 \\ \hline & 0 & 0 & 0 & 0 \end{array} \end{array}$$

$$= \left(\begin{array}{ccc|c} \boxed{1} & 0 & 4 & -5 \\ 0 & \boxed{1} & -3 & 3 \end{array}\right)$$

$$\boxed{x_3 = 1} \Rightarrow \begin{matrix} x_1 = -9 \\ x_2 = 6 \end{matrix}$$

$$= \left(\begin{array}{cc|cc} \boxed{1} & 0 & -4 & -5 \\ 0 & \boxed{1} & +3 & 3 \end{array}\right) \Leftrightarrow \begin{aligned} x_1 &= -5 - 4x_3 \\ x_2 &= 3 + 3x_3 \\ x_3 &= x_3 \in \mathbb{R} \end{aligned}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 - 4x_3 \\ 3 + 3x_3 \\ x_3 + 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}, \ x_3 \in \mathbb{R}}$$