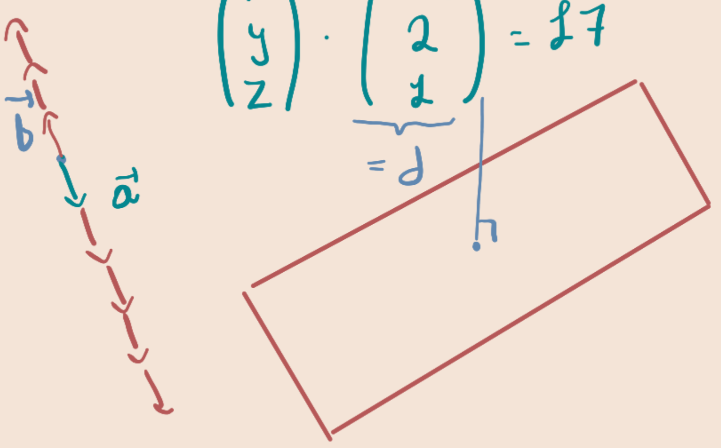
1. A line is given by $\vec{r} = \lambda\vec{a} + \vec{b}$ with $\vec{a} = (1,-1,4)^T$, $\vec{b} = (4,5,6)^T$, while the equation of the plane is given by $-2x+2y+z=17$. Where does the line intersect the plane?

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}}_{=\vec{d}} = 17$$



$$\vec{r} = \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 4+\lambda \\ 5-\lambda \\ 6+4\lambda \end{pmatrix}$$

$$-2(4+\lambda) + 2(5-\lambda) + (6+4\lambda) = 17$$

$$-8 - 2\lambda + 10 - 2\lambda + 6 + 4\lambda = 17$$

$$\Rightarrow \boxed{8 = 17} \ ↯$$

No intersection!

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2. What is the equation of the hyperplane given by $\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \vec{P}_0 + \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$ with $\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$?

$$z - t - x - y = -1 \quad \Rightarrow \quad \boxed{z - t - x - y + 1 = 0}$$

$$= \cancel{\alpha} + \cancel{\beta} + \cancel{\gamma} - \cancel{\alpha} - 1 - \cancel{\beta} - \cancel{\gamma} = -1$$

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \\ 0 \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha+1 \\ \beta \\ \gamma \\ \alpha+\beta+\gamma \end{bmatrix}$$

$$\begin{aligned} t &= \alpha + 1 \\ x &= \beta \\ y &= \gamma \\ z &= \alpha + \beta + \gamma \end{aligned}$$

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = -1$$

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3. Find the cross product of $\vec{u} = \langle 3, 2, -1 \rangle$, $\vec{v} = \langle 1, 1, 0 \rangle$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ — & \vec{u} & — \\ — & \vec{v} & — \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

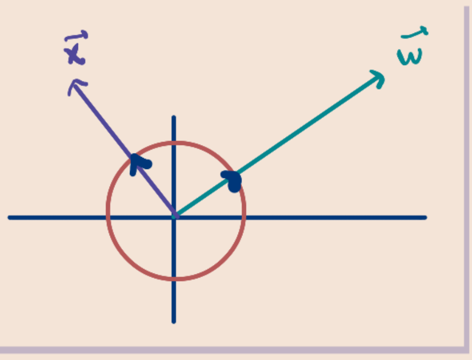
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \hat{i} \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= \hat{i} \cdot 1 - \hat{j} \cdot 1 + \hat{k} \cdot 1$$

$$\boxed{= \hat{i} - \hat{j} + \hat{k} = \langle 1, -1, 1 \rangle}$$

4. Find the unit vector along the direction of $\vec{u}\times\vec{v}$. $\begin{cases}\vec{u} = \langle 7,-1,3\rangle \\ \vec{v} = \langle 2,0,-2\rangle\end{cases}$



Norm (length) of $\vec{w}$

$$\boxed{\|\vec{w}\| = \sqrt{w_1^2+w_2^2+w_3^2}}$$

$$\vec{w}_u = \langle w_1, w_2, w_3\rangle \cdot \frac{1}{\|w\|} = \left\langle \frac{w_1}{\|w\|}, \frac{w_2}{\|w\|}, \frac{w_3}{\|w\|}\right\rangle$$

Claim: $\|\vec{w}_u\| = 1$

Pf. $\|\vec{w}_u\| = \sqrt{\frac{1}{\|w\|^2}(w_1^2+w_2^2+w_3^2)} = \frac{1}{\|w\|}\boxed{\sqrt{(w_1^2+w_2^2+w_3^2)}} = \frac{1}{\cancel{\|w\|}}\cancel{\|w\|} = 1$

$$\vec{u}\times\vec{v} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ — & \vec{u} & — \\ — & \vec{v} & —\end{vmatrix}$$

$$\cdot \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d\end{vmatrix} = ad-bc$$

$$\vec{u}\times\vec{v} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 7 & -1 & 3 \\ 2 & 0 & -2\end{vmatrix} = \hat{i}\begin{vmatrix}-1 & 3\\ 0 & -2\end{vmatrix} - \hat{j}\begin{vmatrix}7 & 3\\ 2 & -2\end{vmatrix} + \hat{k}\begin{vmatrix}7 & -1\\ 2 & 0\end{vmatrix}$$

$$= \hat{i}\cdot 2 + \hat{j}\cdot 20 + \hat{k}\cdot 2 = \langle 2, 20, 2\rangle$$

$$*\|\vec{u}\times\vec{v}\| = \sqrt{2^2+20^2+2^2} = \sqrt{408}$$

$$(\vec{u}\times\vec{v})_u = (\vec{u}\times\vec{v})\frac{1}{\|\vec{u}\times\vec{v}\|} = \boxed{\langle 2, 20, 2\rangle\cdot\frac{1}{\sqrt{408}}}$$

5. Let $\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) = (123),(231),(312) \\ -1 & \text{if } (ijk) = (132),(321),(213) \\ 0 & \text{else}\end{cases}$. What is the k^{th} component of $\vec{u} = \langle u_1, u_2, u_3\rangle \times \vec{v} = \langle v_1, v_2, v_3\rangle$?

$$\vec{u}\times\vec{v} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ — & \vec{u} & — \\ — & \vec{v} & —\end{vmatrix}$$

$$\cdot \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d\end{vmatrix} = ad-bc$$

$$\vec{u}\times\vec{v} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{vmatrix} = \hat{i}(u_2v_3 - v_2u_3) - j(u_1v_3 - v_1u_3) + k(u_1v_2 - u_2v_1)$$

$$= \hat{i}\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix} - j\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix} + k\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}$$

$$= \begin{array}{cl} \begin{bmatrix} +1u_2v_3 - u_3v_2 \\ +1u_3v_1 - u_1v_3 \\ +1u_1v_2 - u_2v_1 \end{bmatrix} & \begin{array}{l} \leftarrow k=1 \\ \leftarrow k=2 \\ \leftarrow k=3 \end{array} \end{array}$$

$$\boxed{[\vec{u}\times\vec{v}]_k = \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ijk}\, u_i v_j}$$

$\boxed{k=1}$

$$\begin{cases} \cancel{111} & \cancel{211} & \cancel{311} \\ \cancel{121} & \cancel{221} & 321 \\ \cancel{131} & 231 & \cancel{331} \end{cases}$$

Exercise. Mark ↗ for zero, ○ for +1, ○ for −1 coefficients!

$\boxed{k=2}$

$$\begin{cases} 112 & 212 & 312 \\ 122 & 222 & 322 \\ 132 & 232 & 332 \end{cases}$$

$\boxed{k=3}$

$$\begin{cases} 113 & 213 & 313 \\ 123 & 223 & 323 \\ 133 & 233 & 333 \end{cases}$$

6. Find a basis for $\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\in\mathbb{R}^3 : 7x+2y-5z=0\right\}$.

$7x+2y-5z=0$, $\quad y=0 \Rightarrow 7x+\overset{=0}{\cancel{2y}}-5z=0 \qquad (x,z)=(5,7)$

$x=0 \Rightarrow \overset{=0}{\cancel{7x}}+2y-5z=0 \qquad (y,z)=(5,2)$

$$\boxed{\left\{\begin{bmatrix}5\\0\\7\end{bmatrix},\begin{bmatrix}0\\5\\2\end{bmatrix}\right\}}$$

$\begin{bmatrix}5\\0\\7\end{bmatrix},\begin{bmatrix}0\\5\\2\end{bmatrix}$ lin. indep. if

$=\vec{0} \Rightarrow \begin{matrix}\alpha_1=0\\ \alpha_2=0\end{matrix}$

~~(⟺)~~

$$\alpha_1\begin{bmatrix}5\\0\\7\end{bmatrix}+\alpha_2\begin{bmatrix}0\\5\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}5\alpha_1\\5\alpha_2\\7\alpha_1+2\alpha_2\end{bmatrix} \begin{matrix}\Rightarrow \alpha_1=0\\ \Rightarrow \alpha_2=0\end{matrix}$$

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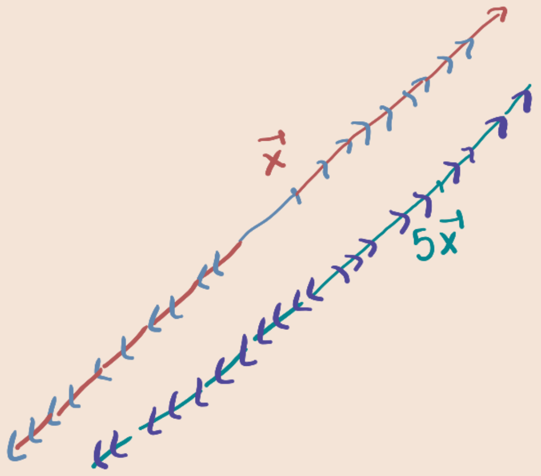
7. Find a basis for $\left\{\begin{bmatrix}3a\\7a\\11a\end{bmatrix}\in\mathbb{R}^3 : a\in\mathbb{R}\right\}$.

$$= \left\{a\underbrace{\begin{bmatrix}3\\7\\11\end{bmatrix}}_{\vec{x}}\in\mathbb{R}^3 : a\in\mathbb{R}\right\}$$

$$= \left\{a\begin{bmatrix}15\\35\\55\end{bmatrix}\in\mathbb{R}^3 : a\in\mathbb{R}\right\}$$

$$B=\left\{\begin{bmatrix}3\\7\\11\end{bmatrix}\right\}$$

$$=\left\{\begin{bmatrix}15\\35\\55\end{bmatrix}\right\}$$



$$\boxed{\begin{matrix}\alpha\cdot\mathbb{R}=\mathbb{R}\\ \underset{\in\mathbb{R}}{\alpha}\cdot\underset{\in\mathbb{R}}{x}\in\mathbb{R}\end{matrix}}$$

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8. What is not a basis for the space of all cubic polynomials?

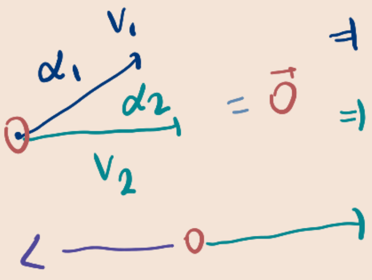
A. $\{x^3, x^2, x, 1\}$ ✓

B. $\{x^3-x^2, x^2-x, x-1, 1\}$ ✓

$\boxed{\text{C. } \{x^3-x^2, x^3-x, x^2-x, x^3-1\}}$

D. $\{x^3+x^2+x+1, (x-6)^2, x-10, 1\}$ ✓

$\{x^3-x^2, x^3-x, x^2-x, x^3-1\}$ is dependent!



$= \vec{0} \quad \begin{matrix}\Rightarrow \alpha_1=0\\ \Rightarrow \alpha_2=0\end{matrix}$

Goal: Cancel out basis vectors (with some scalings)

$$\overset{=1}{\alpha_1}(x^3-x^2)+\overset{=-1}{\alpha_2}(x^3-x)+\overset{=1}{\alpha_3}(x^2-x)+\overset{=0}{\alpha_4}(x^3-1)=0 \not\Rightarrow \begin{cases}\alpha_1=0\\ \alpha_2=0\\ \alpha_3=0\\ \alpha_4=0\end{cases}$$

$$\cancel{x^3}-\cancel{x^2}-\cancel{x^3}+\cancel{x}+\cancel{x^2}-\cancel{x}$$

Reflection...

$$\alpha_1 x^3+\alpha_2 x^2+\alpha_3 x+\alpha_4\cdot 1=0x^3+0x^2+0x+0\cdot 1$$

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

$\alpha_1=0 \quad \alpha_2=0 \quad \alpha_3=0 \quad \alpha_4=0$

Exercise. Check $\{x^3-x^2, x^2-x, x-1, 1\}$ linearly independent!

Mega Exercise. Check $\{x^3+x^2+x+1, (x-6)^2, x-10, 1\}$ linearly independent!

9. Define $v_1 \tilde{+} v_2 = v_1 \cdot v_2$, $c \tilde{\cdot} v_1 = c \cdot v_1$ for scalars $c \in \mathbb{R}$. Is $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ a vector space? $V = \mathbb{R}_+$, $7 \tilde{+} 3 = 7 \cdot 3 = 21$; $1 \tilde{+} 1 = 1 \cdot 1 = 1$

$\boxed{\text{NO}}$

Vector Space checklist.

0. Addition and scalar multiplication are commutative, associative. ☑
1. ~~V is closed under addition and scalar multiplication.~~ ☒
2. There exists a zero vector, neutral element of addition in V ☐
3. Scaling by $1 \in \mathbb{F}$ yields the same vector ☐

$\underbrace{v_1}_{>0} \tilde{+} \underbrace{v_2}_{>0} = \underbrace{v_1}_{>0} \cdot \underbrace{v_2}_{>0} \in V = \mathbb{R}_+$ ✓, $\mathbb{R} \ni \underbrace{c}_{\cancel{>0}} \cdot \underbrace{\underbrace{v_1}_{>0}}_{>0} \notin V = \mathbb{R}_+$

non-ex: $-5 \cdot v_1 \notin \mathbb{R}_+$

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10. Is the set of all degree 10 polynomials with real coefficients a vector space? $\mathbb{R}[x]^{10} := \{a_0 + a_1 x + \dots + a_{10} x^{10} : a_i \in \mathbb{R}, a_{10} \neq 0\}$

[redacted]

$\boxed{\text{NO}}$

$\boxed{\underset{\substack{\uparrow \\ V}}{v_1} + \underset{\substack{\uparrow \\ V}}{v_2} \in V?}$

$$v_1 = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{10} \end{bmatrix}, \quad v_2 = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{10} \end{bmatrix}, \quad (\in \mathbb{R})$$

$$\Rightarrow v_1 + v_2 = \begin{bmatrix} a_0 + b_0 \\ \vdots \\ a_{10} + b_{10} \end{bmatrix}, \quad \underset{\substack{\uparrow \\ \mathbb{R}}}{c_i} = \underset{\in \mathbb{R}}{a_i} + \underset{\in \mathbb{R}}{b_i}$$

$\therefore c_i \in \mathbb{R}$ ✓

$\boxed{c_{10} = a_{10} + b_{10} \neq 0?}$ $a_{10} \neq 0,\ b_{10} \neq 0 \not\Rightarrow c_{10} \neq 0$

ex: $1 + -1 = 0$ $\Leftrightarrow$ $x^{10} + (-x^{10}) = 0 \notin V$