Abstract

Noise sensitivity was first introduced by Benjamini, Kalai and Schramm in their seminal work on Boolean functions. We propose a construction of a similar taste on binary relations. By flipping every relation with a small probability p, a natural question on recoverability arises, to which we give a positive answer in the case of an equivalence relation $(X, \sim)$. We prove that equivalence relations are noise-stable under the prescribed model. In particular, we propose a simple reconstruction algorithm, and show that it achieves an asymptotically zero misclassification error.

Noise Sensitivity on Equivalence Relations

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Constructor University Bremen

StuKon 2025

*Supervised by Prof. Dr. Keivan Mallahi-Karai

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Kahn-Kalai-Linial

$$f(\tilde{x}) \neq f(x) \iff |k_0 - k_1| > |f(x) - \frac{n}{2}|$$

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$$\mathbb{P}[f(x) \neq f(\tilde{x})] \xrightarrow{n \to \infty} \frac{2}{\pi} \arcsin \sqrt{p}$$

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3. If $L = 1$, then $\sim$ is *noise sensitive.*

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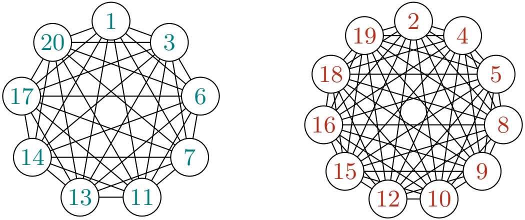

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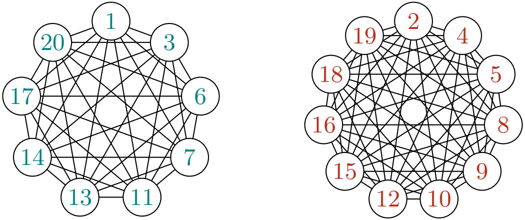

Setup. Inner connectivity, outer isolation.

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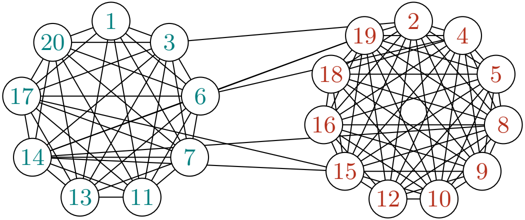

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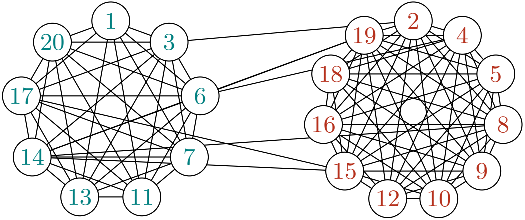

Question. What happened?

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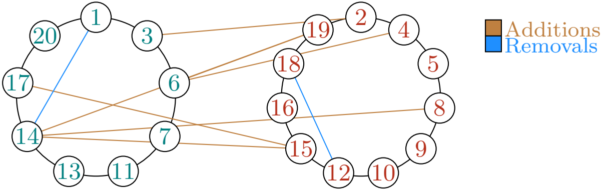

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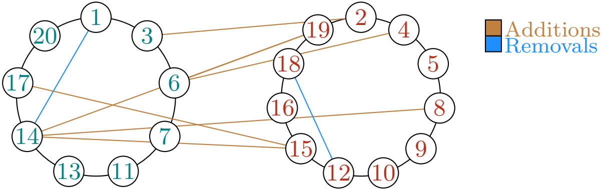

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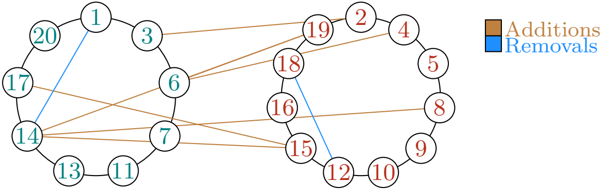

Question. Can we recover $\sim$?

A Proposal

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$i \sim j \wedge j \sim k \Rightarrow i \sim k$

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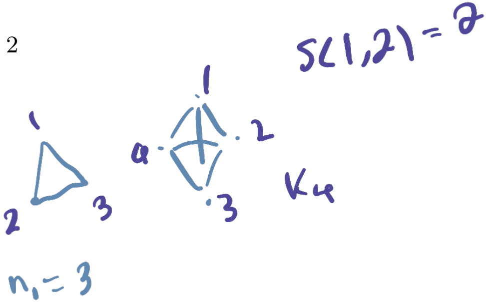

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$$S_X := \begin{pmatrix} s(1,1) & s(1,2) & \dots & s(1,n) \\ s(2,1) & s(2,2) & \dots & s(2,n) \\ \vdots & \dots & \ddots & \vdots \\ s(n,1) & \dots & s(n,n-1) & s(n,n) \end{pmatrix}$$

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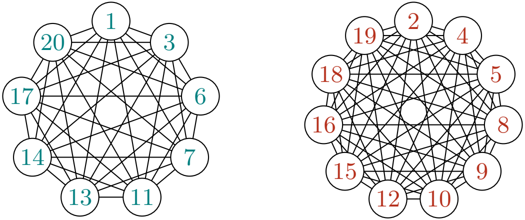

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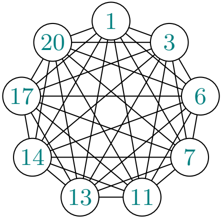

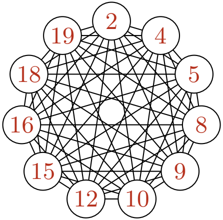

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
8	0	7	0	0	7	7	0	0	0	7	0	7	7	0	0	7	0	0	7	1
0	10	0	9	9	0	0	9	9	9	0	9	0	0	9	9	0	9	9	0	2
7	0	8	0	0	7	7	0	0	0	7	0	7	7	0	0	7	0	0	7	3
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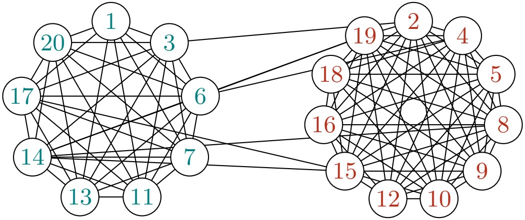

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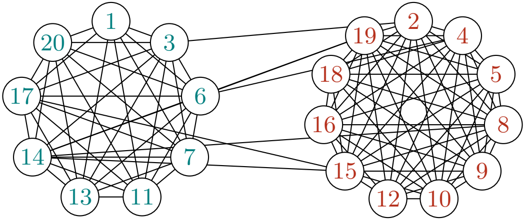

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7	1	6	1	0	6	6	0	0	0	6	0	6	7	1	0	6	0	1	6	1
1	11	0	9	9	3	1	9	9	9	1	8	1	4	9	9	2	8	9	1	2
6	0	9	2	1	7	7	2	1	1	7	1	7	6	3	1	7	1	3	7	3
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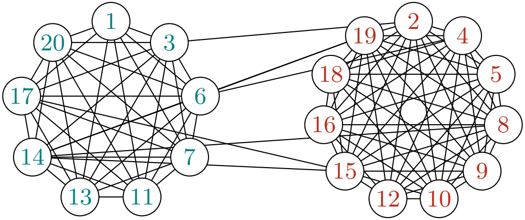

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1	11	0	9	9	3	1	9	9	9	1	8	1	4	9	9	2	8	9	1	| 2
6	0	9	2	1	7	7	2	1	1	7	1	7	6	3	1	7	1	3	7	| 3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Observation.

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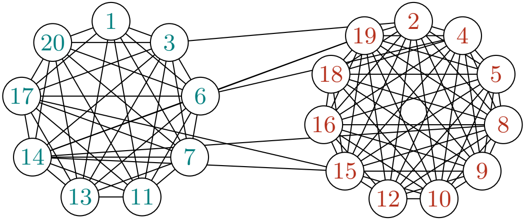


	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	7	1	6	1	0	6	6	0	0	0	6	0	6	7	1	0	6	0	1	6
2	1	11	0	9	9	3	1	9	9	9	1	8	1	4	9	9	2	8	9	1
3	6	0	9	2	1	7	7	2	1	1	7	1	7	6	3	1	7	1	3	7
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Observation. Visible *gap* between same-class & different-class scores.

Experiment

Question.

Experiment

Question. Is this a global phenomena?

Experiment

Question. Is this a global phenomena?

Example (Noisy Equivalence Relation, $p = 1/20$).

Experiment

Question. Is this a global phenomena?

Example (Noisy Equivalence Relation, $p = 1/20$).

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
7	1	6	1	0	6	6	0	0	0	6	0	6	7	1	0	6	0	1	6	1
1	11	0	9	9	3	1	9	9	9	1	8	1	4	9	9	2	8	9	1	2
6	0	9	2	1	7	7	2	1	1	7	1	7	6	3	1	7	1	3	7	3
1	9	2	11	9	1	1	9	9	9	1	8	1	4	9	9	2	8	10	1	4
0	9	1	9	10	2	0	9	9	9	0	8	0	3	9	9	1	8	9	0	5
6	3	7	1	2	10	7	3	2	2	7	2	7	7	4	2	7	2	2	7	6
6	1	7	1	0	7	8	1	0	0	7	0	7	6	2	0	7	0	2	7	7
0	9	2	9	9	3	1	11	9	9	1	8	1	2	10	9	2	8	10	1	8
0	9	1	9	9	2	0	9	10	9	0	8	0	3	9	9	1	8	9	0	9
0	9	1	9	9	2	0	9	9	10	0	8	0	3	9	9	1	8	9	0	10
6	1	7	1	0	7	7	1	0	0	8	0	7	6	2	0	7	0	2	7	11
0	8	1	8	8	2	0	8	8	8	0	9	0	3	8	8	1	9	8	0	12
6	1	7	1	0	7	7	1	0	0	7	0	8	6	2	0	7	0	2	7	13
7	4	6	4	3	7	6	2	3	3	6	3	6	10	3	3	7	3	3	6	14
1	9	3	9	9	4	2	10	9	9	2	8	2	3	12	9	1	8	10	2	15
0	9	1	9	9	2	0	9	9	9	0	8	0	3	9	10	1	8	9	0	16
6	2	7	2	1	7	7	2	1	1	7	1	7	7	1	1	9	1	3	7	17
0	8	1	8	8	2	0	8	8	8	0	9	0	3	8	8	1	9	8	0	18
1	9	3	10	9	2	2	10	9	9	2	8	2	3	10	9	3	8	12	2	19
6	1	7	1	0	7	7	1	0	0	7	0	7	6	2	0	7	0	2	8	20

Observation.

Experiment

Question. Is this a global phenomena?

Example (Noisy Equivalence Relation, $p = 1/20$).

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
7	1	6	1	0	6	6	0	0	0	6	0	6	7	1	0	6	0	1	6	1
1	11	0	9	9	3	1	9	9	9	1	8	1	4	9	9	2	8	9	1	2
6	0	9	2	1	7	7	2	1	1	7	1	7	6	3	1	7	1	3	7	3
1	9	2	11	9	1	1	9	9	9	1	8	1	4	9	9	2	8	10	1	4
0	9	1	9	10	2	0	9	9	9	0	8	0	3	9	9	1	8	9	0	5
6	3	7	1	2	10	7	3	2	2	7	2	7	7	4	2	7	2	2	7	6
6	1	7	1	0	7	8	1	0	0	7	0	7	6	2	0	7	0	2	7	7
0	9	2	9	9	3	1	11	9	9	1	8	1	2	10	9	2	8	10	1	8
0	9	1	9	9	2	0	9	10	9	0	8	0	3	9	9	1	8	9	0	9
0	9	1	9	9	2	0	9	9	10	0	8	0	3	9	9	1	8	9	0	10
6	1	7	1	0	7	7	1	0	0	8	0	7	6	2	0	7	0	2	7	11
0	8	1	8	8	2	0	8	8	8	0	9	0	3	8	8	1	9	8	0	12
6	1	7	1	0	7	7	1	0	0	7	0	8	6	2	0	7	0	2	7	13
7	4	6	4	3	7	6	2	3	3	6	3	6	10	3	3	7	3	3	6	14
1	9	3	9	9	4	2	10	9	9	2	8	2	3	12	9	1	8	10	2	15
0	9	1	9	9	2	0	9	9	9	0	8	0	3	9	10	1	8	9	0	16
6	2	7	2	1	7	7	2	1	1	7	1	7	7	1	1	9	1	3	7	17
0	8	1	8	8	2	0	8	8	8	0	9	0	3	8	8	1	9	8	0	18
1	9	3	10	9	2	2	10	9	9	2	8	2	3	10	9	3	8	12	2	19
6	1	7	1	0	7	7	1	0	0	7	0	7	6	2	0	7	0	2	8	20

Observation. Not very useful.

Experiment

Question. Is this a global phenomena?

Example (Noisy Equivalence Relation, $p = 1/20$).

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
7	1	6	1	0	6	6	0	0	0	6	0	6	7	1	0	6	0	1	6	1
1	11	0	9	9	3	1	9	9	9	1	8	1	4	9	9	2	8	9	1	2
6	0	9	2	1	7	7	2	1	1	7	1	7	6	3	1	7	1	3	7	3
1	9	2	11	9	1	1	9	9	9	1	8	1	4	9	9	2	8	10	1	4
0	9	1	9	10	2	0	9	9	9	0	8	0	3	9	9	1	8	9	0	5
6	3	7	1	2	10	7	3	2	2	7	2	7	7	4	2	7	2	2	7	6
6	1	7	1	0	7	8	1	0	0	7	0	7	6	2	0	7	0	2	7	7
0	9	2	9	9	3	1	11	9	9	1	8	1	2	10	9	2	8	10	1	8
0	9	1	9	9	2	0	9	10	9	0	8	0	3	9	9	1	8	9	0	9
0	9	1	9	9	2	0	9	9	10	0	8	0	3	9	9	1	8	9	0	10
6	1	7	1	0	7	7	1	0	0	8	0	7	6	2	0	7	0	2	7	11
0	8	1	8	8	2	0	8	8	8	0	9	0	3	8	8	1	9	8	0	12
6	1	7	1	0	7	7	1	0	0	7	0	8	6	2	0	7	0	2	7	13
7	4	6	4	3	7	6	2	3	3	6	3	6	10	3	3	7	3	3	6	14
1	9	3	9	9	4	2	10	9	9	2	8	2	3	12	9	1	8	10	2	15
0	9	1	9	9	2	0	9	9	9	0	8	0	3	9	10	1	8	9	0	16
6	2	7	2	1	7	7	2	1	1	7	1	7	7	1	1	9	1	3	7	17
0	8	1	8	8	2	0	8	8	8	0	9	0	3	8	8	1	9	8	0	18
1	9	3	10	9	2	2	10	9	9	2	8	2	3	10	9	3	8	12	2	19
6	1	7	1	0	7	7	1	0	0	7	0	7	6	2	0	7	0	2	8	20

Observation. Not very useful. Sort row values ascendingly.

Experiment

Example (Noisy Equivalence Relation, $p = 1/20$).

Experiment

Example (Noisy Equivalence Relation, $p = 1/20$).

0	0	0	0	0	0	0	1	1	1	1	6	6	6	6	6	6	6	7	7	1
0	1	1	1	1	1	2	3	4	8	8	9	9	9	9	9	9	9	9	11	2
0	1	1	1	1	1	1	2	2	3	3	6	6	7	7	7	7	7	7	9	3
1	1	1	1	1	1	2	2	4	8	8	9	9	9	9	9	9	9	10	11	4
0	0	0	0	0	1	1	2	3	8	8	9	9	9	9	9	9	9	9	10	5
1	2	2	2	2	2	2	2	3	3	4	6	7	7	7	7	7	7	7	10	6
0	0	0	0	0	0	1	1	1	2	2	6	6	7	7	7	7	7	7	8	7
0	1	1	1	1	2	2	2	3	8	8	9	9	9	9	9	9	10	10	11	8
0	0	0	0	0	1	1	2	3	8	8	9	9	9	9	9	9	9	9	10	9
0	0	0	0	0	1	1	2	3	8	8	9	9	9	9	9	9	9	9	10	10
0	0	0	0	0	0	1	1	1	2	2	6	6	7	7	7	7	7	7	8	11
0	0	0	0	0	1	1	2	3	8	8	8	8	8	8	8	8	8	9	9	12
0	0	0	0	0	0	1	1	1	2	2	6	6	7	7	7	7	7	7	8	13
2	3	3	3	3	3	3	3	3	4	4	6	6	6	6	6	7	7	7	10	14
1	1	2	2	2	2	3	3	4	8	8	9	9	9	9	9	9	10	10	12	15
0	0	0	0	0	1	1	2	3	8	8	9	9	9	9	9	9	9	9	10	16
1	1	1	1	1	1	1	2	2	2	3	6	7	7	7	7	7	7	7	9	17
0	0	0	0	0	1	1	2	3	8	8	8	8	8	8	8	8	8	9	9	18
1	2	2	2	2	2	3	3	3	8	8	9	9	9	9	9	10	10	10	12	19
0	0	0	0	0	0	1	1	1	2	2	6	6	7	7	7	7	7	7	8	20

Experiment

Example (Noisy Equivalence Relation, $p = 1/20$).

$$
\begin{pmatrix}
0&0&0&0&0&0&0&1&1&1&1&6&6&6&6&6&6&6&7&7\\
0&1&1&1&1&1&2&3&4&8&8&9&9&9&9&9&9&9&9&11\\
0&1&1&1&1&1&1&2&2&3&3&6&6&7&7&7&7&7&7&9\\
1&1&1&1&1&1&2&2&4&8&8&9&9&9&9&9&9&9&10&11\\
0&0&0&0&0&1&1&2&3&8&8&9&9&9&9&9&9&9&9&10\\
1&2&2&2&2&2&2&2&3&3&4&6&7&7&7&7&7&7&7&10\\
0&0&0&0&0&0&1&1&1&2&2&6&6&7&7&7&7&7&7&8\\
0&1&1&1&1&2&2&2&3&8&8&9&9&9&9&9&9&10&10&11\\
0&0&0&0&0&1&1&2&3&8&8&9&9&9&9&9&9&9&9&10\\
0&0&0&0&0&1&1&2&3&8&8&9&9&9&9&9&9&9&9&10\\
0&0&0&0&0&0&1&1&1&2&2&6&6&7&7&7&7&7&7&8\\
0&0&0&0&0&1&1&2&3&8&8&8&8&8&8&8&8&8&9&9\\
0&0&0&0&0&0&1&1&1&2&2&6&6&7&7&7&7&7&7&8\\
2&3&3&3&3&3&3&3&3&4&4&6&6&6&6&6&7&7&7&10\\
1&1&2&2&2&2&3&3&4&8&8&9&9&9&9&9&9&10&10&12\\
0&0&0&0&0&1&1&2&3&8&8&9&9&9&9&9&9&9&9&10\\
1&1&1&1&1&1&1&2&2&2&3&6&7&7&7&7&7&7&7&9\\
0&0&0&0&0&1&1&2&3&8&8&8&8&8&8&8&8&8&9&9\\
1&2&2&2&2&2&3&3&3&8&8&9&9&9&9&9&10&10&10&12\\
0&0&0&0&0&0&1&1&1&2&2&6&6&7&7&7&7&7&7&8
\end{pmatrix}
\begin{array}{|l}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20
\end{array}
$$

Observation.

Experiment

Example (Noisy Equivalence Relation, $p = 1/20$).

$$\begin{pmatrix}
0&0&0&0&0&0&0&1&1&1&1&6&6&6&6&6&6&6&7&7\\
0&1&1&1&1&1&2&3&4&8&8&9&9&9&9&9&9&9&9&11\\
0&1&1&1&1&1&1&2&2&3&3&6&6&7&7&7&7&7&7&9\\
1&1&1&1&1&1&2&2&4&8&8&9&9&9&9&9&9&9&10&11\\
0&0&0&0&0&1&1&2&3&8&8&9&9&9&9&9&9&9&9&10\\
1&2&2&2&2&2&2&2&3&3&4&6&7&7&7&7&7&7&7&10\\
0&0&0&0&0&0&1&1&1&2&2&6&6&7&7&7&7&7&7&8\\
0&1&1&1&1&2&2&2&3&8&8&9&9&9&9&9&9&10&10&11\\
0&0&0&0&0&1&1&2&3&8&8&9&9&9&9&9&9&9&9&10\\
0&0&0&0&0&1&1&2&3&8&8&9&9&9&9&9&9&9&9&10\\
0&0&0&0&0&0&1&1&1&2&2&6&6&7&7&7&7&7&7&8\\
0&0&0&0&0&1&1&2&3&8&8&8&8&8&8&8&8&8&9&9\\
0&0&0&0&0&0&1&1&1&2&2&6&6&7&7&7&7&7&7&8\\
2&3&3&3&3&3&3&3&3&4&4&6&6&6&6&6&7&7&7&10\\
1&1&2&2&2&2&3&3&4&8&8&9&9&9&9&9&9&10&10&12\\
0&0&0&0&0&1&1&2&3&8&8&9&9&9&9&9&9&9&9&10\\
1&1&1&1&1&1&1&2&2&2&3&6&7&7&7&7&7&7&7&9\\
0&0&0&0&0&1&1&2&3&8&8&8&8&8&8&8&8&8&9&9\\
1&2&2&2&2&2&3&3&3&8&8&9&9&9&9&9&10&10&10&12\\
0&0&0&0&0&0&1&1&1&2&2&6&6&7&7&7&7&7&7&8
\end{pmatrix}
\begin{matrix}
|\,1\\|\,2\\|\,3\\|\,4\\|\,5\\|\,6\\|\,7\\|\,8\\|\,9\\|\,10\\|\,11\\|\,12\\|\,13\\|\,14\\|\,15\\|\,16\\|\,17\\|\,18\\|\,19\\|\,20
\end{matrix}$$

Observation. There is a global phenomena!

Experiment

Example (Noisy Equivalence Relation, $p = 1/20$).

0	0	0	0	0	0	0	1	1	1	1	6	6	6	6	6	6	6	7	7	1
0	1	1	1	1	1	2	3	4	8	8	9	9	9	9	9	9	9	9	11	2
0	1	1	1	1	1	1	2	2	3	3	6	6	7	7	7	7	7	7	9	3
1	1	1	1	1	1	2	2	4	8	8	9	9	9	9	9	9	9	10	11	4
0	0	0	0	0	1	1	2	3	8	8	9	9	9	9	9	9	9	9	10	5
1	2	2	2	2	2	2	2	3	3	4	6	7	7	7	7	7	7	7	10	6
0	0	0	0	0	0	1	1	1	2	2	6	6	7	7	7	7	7	7	8	7
0	1	1	1	1	2	2	2	3	8	8	9	9	9	9	9	9	10	10	11	8
0	0	0	0	0	1	1	2	3	8	8	9	9	9	9	9	9	9	9	10	9
0	0	0	0	0	1	1	2	3	8	8	9	9	9	9	9	9	9	9	10	10
0	0	0	0	0	0	1	1	1	2	2	6	6	7	7	7	7	7	7	8	11
0	0	0	0	0	1	1	2	3	8	8	8	8	8	8	8	8	8	9	9	12
0	0	0	0	0	0	1	1	1	2	2	6	6	7	7	7	7	7	7	8	13
2	3	3	3	3	3	3	3	3	4	4	6	6	6	6	6	7	7	7	10	14
1	1	2	2	2	2	3	3	4	8	8	9	9	9	9	9	9	10	10	12	15
0	0	0	0	0	1	1	2	3	8	8	9	9	9	9	9	9	9	9	10	16
1	1	1	1	1	1	1	2	2	2	3	6	7	7	7	7	7	7	7	9	17
0	0	0	0	0	1	1	2	3	8	8	8	8	8	8	8	8	8	9	9	18
1	2	2	2	2	2	3	3	3	8	8	9	9	9	9	9	10	10	10	12	19
0	0	0	0	0	0	1	1	1	2	2	6	6	7	7	7	7	7	7	8	20

Observation. There is a global phenomena!

Formalism.

Experiment

Example (Noisy Equivalence Relation, $p = 1/20$).

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 7 & 7 \\
0 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 11 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 9 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 4 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 10 & 11 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 10 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 4 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 8 \\
0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 10 & 10 & 11 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 10 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 8 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 8 \\
2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 10 \\
1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 4 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 10 & 10 & 12 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 10 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 9 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 \\
1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 10 & 10 & 10 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 8
\end{pmatrix}
\begin{array}{|l}
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20
\end{array}$$

Observation. There is a global phenomena!

Formalism. What is the expected score gap?

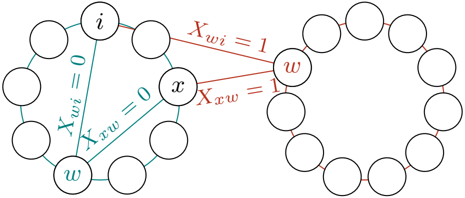
Expected in-Score

Fix $x \in X_i$. If $x \sim i$, then

Expected in-Score

Fix $x \in X_i$. If $x \sim i$, then

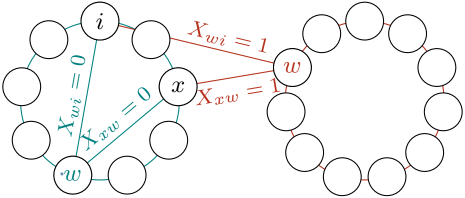
$$\tilde{s}(x,i) = \overbrace{\sum_{w \in X_i \setminus \{x,i\}} (1 - X_{xw}) \cdot (1 - X_{wi})}^{\text{Inter−connections}} + \underbrace{\sum_{w \notin X_i} X_{xw} \cdot X_{wi}}_{\text{Intra−connections}}$$



Expected in-Score

Fix $x \in X_i$. If $x \sim i$, then

$$\tilde{s}(x,i) = \underbrace{\sum_{w \in X_i \setminus \{x,i\}} (1 - X_{xw}) \cdot (1 - X_{wi})}_{\text{Inter}-\text{connections}} + \underbrace{\sum_{w \notin X_i} X_{xw} \cdot X_{wi}}_{\text{Intra}-\text{connections}}$$



Implication. The expected in-score is of order $\mathcal{O}(n_i)$.

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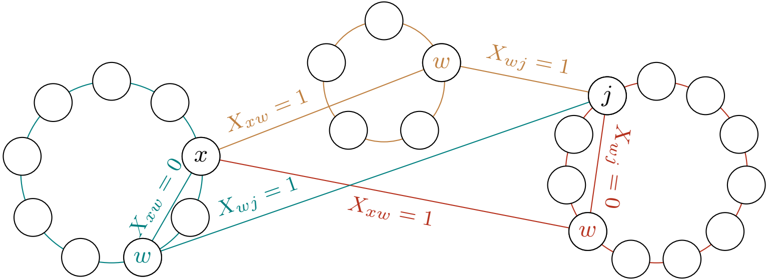

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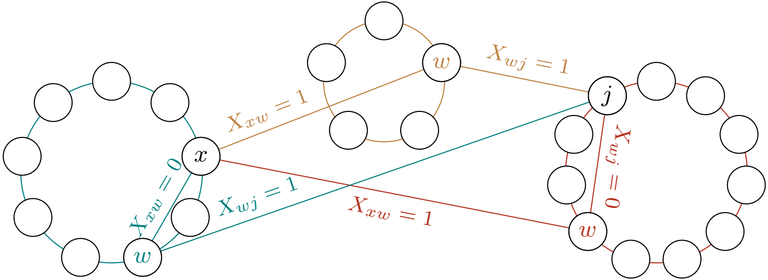

Implication. The expected out-score is of order $\mathcal{O}((n_i + n_j) \cdot p)$.

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If ξ is the expectation of $\hat{\xi}$, then

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Implication. The gap gets larger!

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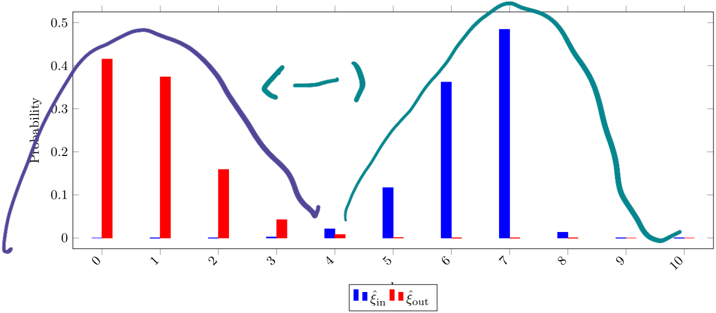

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Implication. To asymptotically bound the deviation probability to zero, demand $\sigma^2 = o(t^2)$.

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Bounding Deviations

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If $\sigma^2 = o(t^2)$, we are done. Using earlier computations yields

$$\begin{aligned}\mathrm{Var}[\hat{\xi}] &= (3n_i + n_j - 6) \cdot p \\ &+ (2n - 11n_i - 5n_j + 18) \cdot p^2 \\ &+ (12n_i + 8n_j - 2n - 20) \cdot p^3 \\ &+ (-4n_i - 4n_j + 8) \cdot p^4 \\ &= \mathcal{O}((n_i + n_j) \cdot p) \qquad (!)\end{aligned}$$

Observation. With $p = \mathcal{O}(\frac{1}{n})$, we get that $\sigma^2 = \mathcal{O}(1)$ is finite!

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- Let q be the probability that one pair of scores ξ_{in}, ξ_{out} makes a deviation of $t = \alpha \cdot \xi$ below ξ. Then,

$$q \leq \overbrace{(n \times n)^2}^{\text{Possible Pairs}} \exp\left(-\frac{\alpha^2 \xi^2}{2\sigma^2 + 2/3 \cdot \alpha \cdot \xi}\right) \qquad \text{(Union Bound)}$$

$$\sim n^4 \exp(-n) \underset{n \to \infty}{\longrightarrow} 0. \qquad \boxed{\xi}$$

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REFERENCES

[1] Itai Benjamini, Gil Kalai, and Oded Schramm. *Noise Sensitivity of Boolean Functions & Applications to Percolation.* 2000. arXiv: math/9811157 [math.PR].

[2] Encyclopedia of Mathematics. *Bernstein inequality.* http://encyclopediaofmath.org/index.php?title=Bernstein_inequality&oldid=15217.

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Thanks!

Try it out

Bachelor's Thesis

Jupyter Notebook