An Algebraic Construction of Complete Regular Maps via Prime Ideals

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Complete Regular Maps (CRMs)

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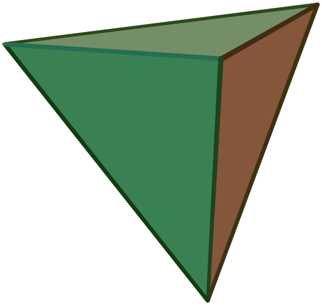
Example

A tetrahedron can be interpreted as the embedding of the complete graph K_4 on 4 vertices into the sphere.

Pictures



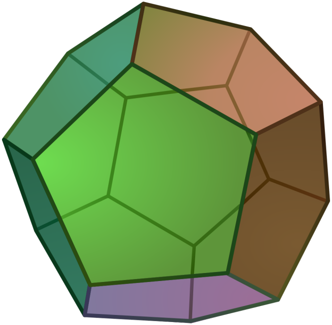
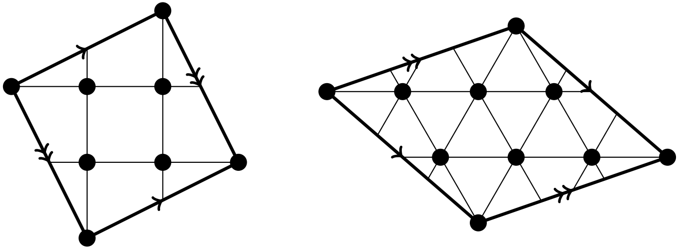


Figure: The tetrahedron is a CRM, the dodecahedron is regular but not complete.

(Figures stolen from Wikipedia.)

Key examples

We obtain CRMs on K_5 and K_7 in the torus by quotienting out the Gaussian and Eisenstein integers by prime ideals.



(a) $K_5 \hookrightarrow \Sigma_1$ by $\mathbb{Z}[i]/(1+2i)$

(b) $K_7 \hookrightarrow \Sigma_1$ by $\mathbb{Z}[\omega]/(3+\omega)$

Figure: Two complete regular maps into surfaces of genus one

Motivation

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A complete regular map with n vertices exists if and only if n is a prime power.

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Question

Can we generalize these constructions to obtain CRMs on $n = p^f$ vertices?

The construction

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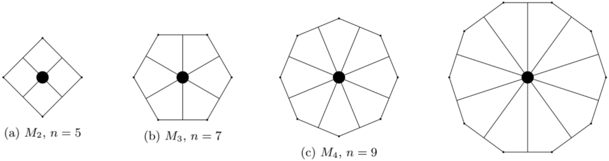
2. Construct a homomorphism $\varphi : \pi_1(X) \to \mathbb{Z}[\zeta_{n-1}]$.
3. For a prime $\mathfrak{p} \subseteq \mathbb{Z}[\zeta_{n-1}]$ containing p, $\varphi^{-1}(\mathfrak{p})$ is an index n subgroup of $\pi_1(X)$.
4. Lift loops in X along a covering map $Y \to X$ arising from this subgroup to get an embedding of a graph in Y.



(a) M_2, $n = 5$ (b) M_3, $n = 7$ (c) M_4, $n = 9$ (d) M_5, $n = 11$

A question

Moral of the story

We have a machine taking a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_{n-1}]$ containing p to a CRM $M_{\mathfrak{p}}$ with n vertices.

The first step in our work was to show this machine really does give us a CRM on n vertices.

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Answer

Yeah.

Which CRMs does this give us?

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Let p be an odd prime and set $n = p^f$. Then there are $\phi(n-1)/f$ prime ideals of $\mathbb{Z}[\zeta_{n-1}]$ containing p.

Classification of (most) CRMs

Distinct prime ideals give distinct maps, so:

Theorem

The function $\mathfrak{p} \mapsto M_{\mathfrak{p}}$ *is a bijection between prime ideals of* $\mathbb{Z}[\zeta_{n-1}]$ *containing* p *and CRMs (up to isomorphism).*

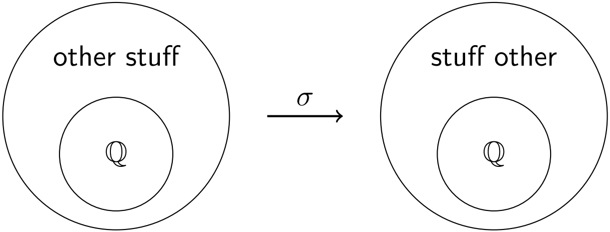
Something Else to Study

Question

What is the action of the Galois group on CRMs?

What is the Galois Group?

The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ is the group of automorphisms $\mathbb{Q}(\zeta_{n-1}) \to \mathbb{Q}(\zeta_{n-1})$ fixing the elements of $\mathbb{Q}$. Visually, it can be represented like so:



What are dessins?

Definition

A K_n-**dessin** is a topological map whose underlying graph is bipartite with n vertices on each side.

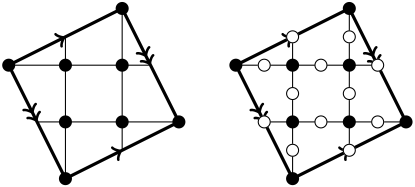


Figure: Bipartification of a CRM to obtain a dessin

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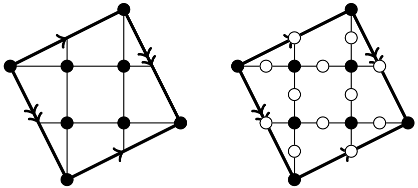

Figure: Bipartification of a CRM to obtain a dessin

K_n-dessins D give surfaces defined by polynomials over $\mathbb{Q}(\zeta_{n-1})$, which yields an action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ on K_n-dessins that we denote by D^σ for $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$.

Our results

Theorem

Given $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ *and a prime* $\mathfrak{p} \subseteq \mathbb{Z}[\zeta_{n-1}]$ *containing* p*, there is an isomorphism*

$$D_{\mathfrak{p}}^{\sigma} \simeq D_{\sigma\mathfrak{p}}$$

of K_n*-dessins.*

So this tells us the two actions of $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ on K_n-dessins are “equivalent”.

Action I of the Galois group

Recall:

- Prime ideals $\mathfrak{p} \subseteq \mathbb{Z}[\zeta_{n-1}]$ give CRMs $M_{\mathfrak{p}}$.
- These $M_{\mathfrak{p}}$ induce K_n-dessins $D_{\mathfrak{p}}$.
- K_n-dessins $D_{\mathfrak{p}}$ give rise to surfaces that can be described by algebraic equations over $\mathbb{Q}(\zeta_{n-1})$.

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- K_n-dessins $D_\mathfrak{p}$ give rise to surfaces that can be described by algebraic equations over $\mathbb{Q}(\zeta_{n-1})$.

$\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ acts on K_n-dessins by acting on the coefficients of these equations.

Denote the action of $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ on $D_\mathfrak{p}$ by $D_\mathfrak{p}^\sigma$).

Action II of the Galois group

- $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ also permutes prime ideals of $\mathbb{Z}[\zeta_{n-1}]$.
- We saw earlier that prime ideals are in bijection with CRMs on n vertices.

Thus, we obtain a second action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ on K_n-dessins: $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ takes $D_{\mathfrak{p}}$ to $D_{\sigma\mathfrak{p}}$.

Statement & Proof Sketch

Theorem

Given $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ and a prime $\mathfrak{p} \subseteq \mathbb{Z}[\zeta_{n-1}]$ containing p, there is an isomorphism

$$D_{\mathfrak{p}}^{\sigma} \simeq D_{\sigma\mathfrak{p}}$$

of K_n-dessins.

Each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ gives rise to an operation H_j on dessins (called a Wilson operator).

Proof.

1. Jones, Streit & Wolfart (2009) proved $D_{\mathfrak{p}}^{\sigma} \simeq H_j D_{\mathfrak{p}}$.
2. We proved $H_j D_{\mathfrak{p}} \simeq D_{\sigma\mathfrak{p}}$ using additonal results from our construction.

Thus $D_{\mathfrak{p}}^{\sigma} \simeq D_{\sigma\mathfrak{p}}$. □

Acknowledgments

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