Mathematical Modeling Lab @ Constructor University

Problem 2

June 15, 2024

A Prelude

A joint-work between Omar Elshinawy & Mohammad Habibi-Bennani, this project was completed in Spring of 2024 for a class in Mathematical Modeling at Constructor University, under Professor Ivan Ovsyannikov and Mr Dzmitry Rumiantsau.

We start by importing the very basic libraries for this project.

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

Problem 2. We set $\zeta = 0.25$, and proceed with $\dot{x} = v$

$$\ddot{x} + 0.5\dot{x} + x = 0$$

which once again yields

$$\frac{d}{dt}\,y := \frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{cases} \dot{x} = v \\ \dot{v} = -0.5v - x \end{cases}$$

that is a first-order system of equations. We can represent this in python as follows,

[2]:
```
def damped_oscillator_2(t, y):
    x, v = y
    return [v, -0.5*v - x]
```

given

$$x(0) = 1, \quad x(9) = 0$$

[3]:
```
x_0 = 1; x_9 = 0
```

and a timeframe

$$t \in [0, 9].$$

[4]:
```
tframe = (0,9)
```

2.1 The Shooting Method.

This method is useful for boundary value problems like our problem here. The idea is to use a guess for the initial velocity v_0, coupled with the given $x(0) = 1$ to attempt solving the system as an intermediate value problem. Here is the recipe:

1. shoot for a guess $v(0)$;
2. solve the IVP using $x(0), v(0)$;
3. check the solution $\tilde{x}(9)$ versus $x(9)$;

if $\tilde{x}(9)$ does not meet the boundary condition $x(9)$ then clearly it is incorrect so adjust the guess and re-iterate.

We define the shooting function for an initial guess v_0, and solve it as an intermediate value problem. We use the solver `solve_ivp` specifically for that purpose.

[5]:
```
from scipy.integrate import solve_ivp
```

Let us now rigorously formulate the previous ideas.

[6]:
```
def shooting_function(v_0):
    sol = solve_ivp(damped_oscillator_2, tframe, [x_0, v_0], t_eval=np.
 ↪linspace(0, 9, 1000))

    # 1. sol.y[0, t] gives x(t)
    # 2. sol.y[0, -1] gives x(9), as -1 returns the last value in the time array

    return sol.y[0, -1] - x_9
```

For the guess v_0, our function returns the value of

$$\tilde{x}(9) - x(9).$$

We want this to be zero. This is due to the fact that

$$x(9) = x(9) \quad \text{if and only if} \quad \tilde{x}(9) - x(9) = 0.$$

Let us worry less about solving this equation. For this purpose, we summon the `root_scalar` method to find us the right guess.

[7]:
```
from scipy.optimize import root_scalar

# Find the guess v_0 which makes the equation 0.
sol = root_scalar(shooting_function, bracket=[-10, 10], method='brentq')
v_0 = sol.root

print(f"v_0 = {v_0}")
```

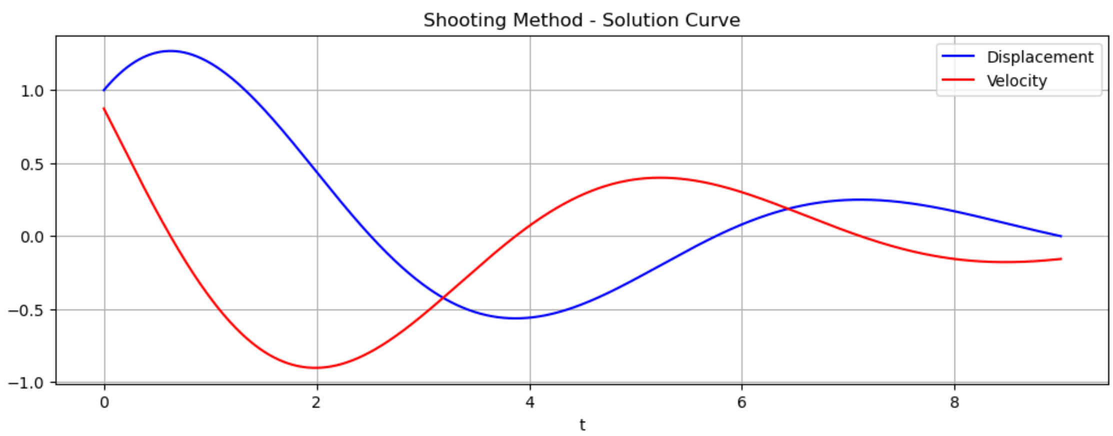
```
v_0 = 0.8741061753595658
```

We can now use the correct v_0 to solve the Intermediate Value Problem.

[8]:
```
# Solve the IVP with the initial displacement + correct initial velocity
sol = solve_ivp(damped_oscillator_2, tframe, [x_0, v_0],  t_eval=np.linspace(0,␣
↪9, 1000))

# extract time values
t_shooting = sol.t

#extract x values
x_shooting = sol.y[0]

#extract v values
v_shooting = sol.y[1]

# Check out the shooting result!
#print(sol)
```

It now remains to "shoot our shot" with the results.

[9]:
```
plt.figure(figsize=(12, 4,))

plt.plot(t_shooting, x_shooting, color='blue', label='Displacement')
plt.plot(t_shooting, v_shooting, color='red', label='Velocity')

plt.xlabel('t')
plt.title('Shooting Method - Solution Curve')
plt.legend()
plt.grid(True)
plt.show()
```



Bullseye. □

2.2 The Finite Difference Method.

0) For our differential equation

$$\ddot{x} + 0.5\dot{x} + x = 0$$

with boundary conditions \$ x(0) = 1 ~and ~ x(9) = 0,\$ the goal is to summon the power of Linear Algebra to find values that approximate the function $x(t)$. * We discretize the time domain $[0, 9]$ into n partitions t_i.
* With n such t_i at hand, we have n unknown variables $x_i := x(t_i)$ which we want to find the value for.
* Therefore, it only makes sense to demand n (linearly independent) equations in n variables.

We show that each equation is of the form

$$0 \cdot x_0 + \cdots + 0 \cdot x_{i-2} + \Big(\frac{1}{h^2} - \frac{0.25}{h}\Big) \cdot x_{i-1} + \Big(\frac{-2}{h^2} + 1\Big) \cdot x_i + \Big(\frac{1}{h^2} + \frac{0.25}{h}\Big) \cdot x_{i+1} + 0 \cdot x_{i+2} + \cdots + 0 \cdot x_n = 0$$

where x_i are the displacement at time t_i. The authors personally think this method is very creative in the way these equations are obtained. Not to cause confusion, we will use the following definitions interchangeabely.

$$\begin{aligned} t_{i\pm 1} := t_i \pm h \quad & \& \quad x_i := x(t_i) \\ \dot{x}_i := \dot{x}(t_i) \quad & \& \quad \ddot{x}_i := \ddot{x}(t_i) \end{aligned}$$

I) We start by discretizing the time domain. Divide the interval $[0, 9]$ into $N + 1$ subintervals to get N interior points. The time step h is therefore

$$h = \frac{t_{N+1} - t_0}{N + 1} := \frac{9}{N + 1}.$$

Thus, the total number of points, including the boundary points, will be $N + 2$. This is okay, however, as we already know the values of x_0 and x_{N+1} as initial conditions. This is consistent with the N unknown variables x_i for $i \in \{1, \ldots, N\}$, which are the interior points. We shall reference them using the same term in the code.

II) We recall that the Taylor expansion of $x(t_i \pm h)$ is given by

$$x(t_{i+1}) := x(t_i + h) = x(t_i) + h\dot{x}(t_i) + \frac{h^2}{2}\ddot{x}(t_i) + \mathcal{O}(h^3) \quad (1) x(t_{i-1}) := x(t_i - h) = x(t_i) - h\dot{x}(t_i) + \frac{h^2}{2}\ddot{x}(t_i) + \mathcal{O}(h^3) \quad (2)$$

(1) + (2) is surprisingly not 3, but it yields (3) as we see below.

$$\ddot{x}_i \approx \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2} \quad (3)$$

More surprising is the fact that $(1)-(2)$ indeed gives (-1), that is the last equation (4) in our array of equations for this construction.

$$\dot{x}_i \approx \frac{x_{i+1}-x_{i-1}}{2h} \quad (4)$$

III) Now, recall that the differential equation was given by $\ddot{x}+0.5\dot{x}+x=0$ so we can use $(3),(4)$ to approximate the solution of $\ddot{x}+0.5\dot{x}+x=0$.

$$\frac{x_{i+1}-2x_i+x_{i-1}}{h^2}+0.5\cdot\frac{x_{i+1}-x_{i-1}}{2h}+x_i=0.$$

With some simple algebra, we obtain that

$$\big(\frac{1}{h^2}-\frac{0.25}{h}\big)\cdot x_{i-1}+\big(\frac{-2}{h^2}+1\big)\cdot x_i+\big(\frac{1}{h^2}+\frac{0.25}{h}\big)\cdot x_{i+1}=0$$

which are all the coefficients we need. Notice that we can write

$$0\cdot x_1+\cdots+0\cdot x_{i-2}+\big(\frac{1}{h^2}-\frac{0.25}{h}\big)\cdot x_{i-1}+\big(\frac{-2}{h^2}+1\big)\cdot x_i+\big(\frac{1}{h^2}+\frac{0.25}{h}\big)\cdot x_{i+1}+0\cdot x_{i+2}+\cdots+0\cdot x_n=0$$

so that in each row only three coeffients are non-zero, namely $x_{i-1}, x_i,\ x_{i+1}$. This makes the coefficient matrix A tri-diagonal, which is crucial for how we construct $A\in\mathbb{R}^{N\times N}$ in our code.

IV) Finally, notice that the variables x_0, x_{N+1} are included in the first and last equations `b[0]` and `b[-1]`. The catch is that our system is defined for $x_1,\dots,x_n$. We need all equations to be of the form

$$a_1x_1+a_2x_2+\cdots+a_nx_n=b_k.$$

Let us look at the two equations that include x_0, x_{N+1}. First,

$$\big(\frac{1}{h^2}-\frac{0.25}{h}\big)\cdot x_0+\big(\frac{-2}{h^2}+1\big)\cdot x_1+\big(\frac{1}{h^2}+\frac{0.25}{h}\big)\cdot x_2=0$$

is equivalent to

$$\big(\frac{-2}{h^2}+1\big)\cdot x_1+\big(\frac{1}{h^2}+\frac{0.25}{h}\big)\cdot x_2=-\big(\frac{1}{h^2}-\frac{0.25}{h}\big)\cdot x_0. \quad (a)$$

Similarly, we get that

$$\big(\frac{1}{h^2}-\frac{0.25}{h}\big)\cdot x_{N-1}+\big(\frac{-2}{h^2}+1\big)\cdot x_N+\big(\frac{1}{h^2}+\frac{0.25}{h}\big)\cdot x_{N+1}=0$$

becomes

$$\big(\frac{1}{h^2}-\frac{0.25}{h}\big)\cdot x_{N-1}+\big(\frac{-2}{h^2}+1\big)\cdot x_N=-\big(\frac{1}{h^2}+\frac{0.25}{h}\big)\cdot x_{N+1}. \quad (b)$$

This is indeed successful, as we can make use of the boundary conditions $x_0,\ x_{N+1}$ which are (respectively) $x(0)=1, x(9)=0$ in our problem here to compute `b[0]` and `b[-1] = b[N]` without any ambiguity.

[10]:
```python
def finite_difference(N, t_span, x_0, x_n1, zeta=0.25):

    # initialising
    h = (t_span[1] - t_span[0]) / (N + 1)
    t = np.linspace(t_span[0], t_span[1], N + 2)
    # matrix A
    A = np.zeros((N, N))
    # n-dimensional solution vector
    b = np.zeros(N)

    # for each row (equation)
    for i in range(N):
        # avoids out of bounds error
        if i > 0:
            A[i, i-1] = 1 / h**2 - 0.25 / h # insert coefficient of x_{i-1}
        A[i, i] = -2 / h**2 + 1 # insert coefficient of {x_i}
        # avoids out of bounds error
        if i < N - 1:
            A[i, i+1] = 1 / h**2 + 0.25 / h # insert coefficient of x_{i+1}

    # treating edge solutions
    b[0]  -= x_0 * (1/h**2 - 0.25/h)
    b[-1] -= x_n1 *(1/h**2 + 0.25/h)

    x_interior = np.linalg.solve(A, b)
    x = np.concatenate([[x_0], x_interior, [x_n1]])
    return t, x
```

We have successfully derived the finite differencing method, so we might as well put this nice construction into immediate use. We set the number of partitions,

[11]:
```python
N = 100
```

then simply call the function with the correct parameters.

[12]:
```python
t_finite_diff, x_finite_diff = finite_difference(N, tframe, x_0, x_9)
```

To get more creative, we use the values `x_finite_diff` to find approximations to $v(t)$. This is made precise by means of equation (4),

$$\dot{x}_i \approx \frac{x_{i+1} - x_{i-1}}{2h} \quad (4)$$

as well as two more equations, namely the notorious

$$\dot{x}_i \approx \frac{x_(t_i + h) - x(t_i)}{h} =: \frac{x_{i+1} - x_i}{h} \quad (5)$$

not to mention the less notorious but fairly obvious

$$\dot{x}_i \approx \frac{x_(t_i) - x(t_i - h)}{h} =: \frac{x_i - x_{i-1}}{h} \quad (6).$$

This is merely a change of variable so to say.

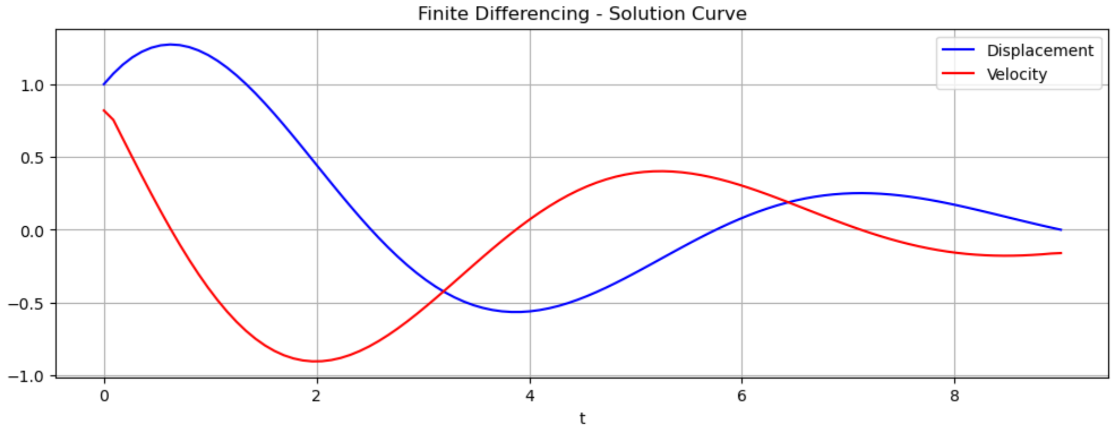
[13]:
```
def compute_velocity(t_finite_diff, x_finite_diff):

    # Timestep is fixed
    h = t_finite_diff[1] - t_finite_diff[0]

    # Give it structure like x_finite diff
    v_finite_diff = np.zeros_like(x_finite_diff)

    # Equation (4) for interior points
    v_finite_diff[1:-1] = (x_finite_diff[2:] - x_finite_diff[:-2]) / (2 * h)

# Treating Edge Solutions #

    # Equation (5) for the first point
    v_finite_diff[0] = (x_finite_diff[1] - x_finite_diff[0]) / h

    # Equation (6) for the last point
    v_finite_diff[-1] = (x_finite_diff[-1] - x_finite_diff[-2]) / h

    return v_finite_diff
```

Now let us derive the array of values for v,

[14]:
```
v_finite_diff = compute_velocity(t_finite_diff, x_finite_diff)
```

Finally, a visually aesthetic curve of displacement versus time.

[15]:
```
plt.figure(figsize=(12, 4,))

plt.plot(t_finite_diff, x_finite_diff, label='Displacement', color='blue')
plt.plot(t_finite_diff, v_finite_diff, label='Velocity', color='red')
plt.xlabel('t')
plt.title('Finite Differencing - Solution Curve')
plt.legend()
plt.grid(True)
plt.show()
```



Ladies and Gentlemen, Finite differencing! □

2.3 `scipy.integrate.solve_bvp`

This is fairly straightforward.

[16]:
```
from scipy.integrate import solve_bvp
```

Start by discretize the interval $[0, 9]$. Let us not forget to initialise the solution array `y = [x, t]` with trivial values as well.

[17]:
```
t = np.linspace(0, 9, 100); y = np.zeros((2, t.size))
```

Recall the initial conditions $x(0) = 1$ & $x(9) = 0$. We pass them down in the form

$$x(0) - 1 = 0 \quad \& \quad x(9) - 0 = 0.$$

Similar to the shooting function, the task is to minimise this residual of functions all the way to zero, so that the solution $\tilde{x}(0) = x(0)$.

[18]:
```
sol = solve_bvp(damped_oscillator_2,
                lambda y_0, y_n: [y_0[0] - 1, y_n[0] - 0],
                t, y)
#print(sol)
```

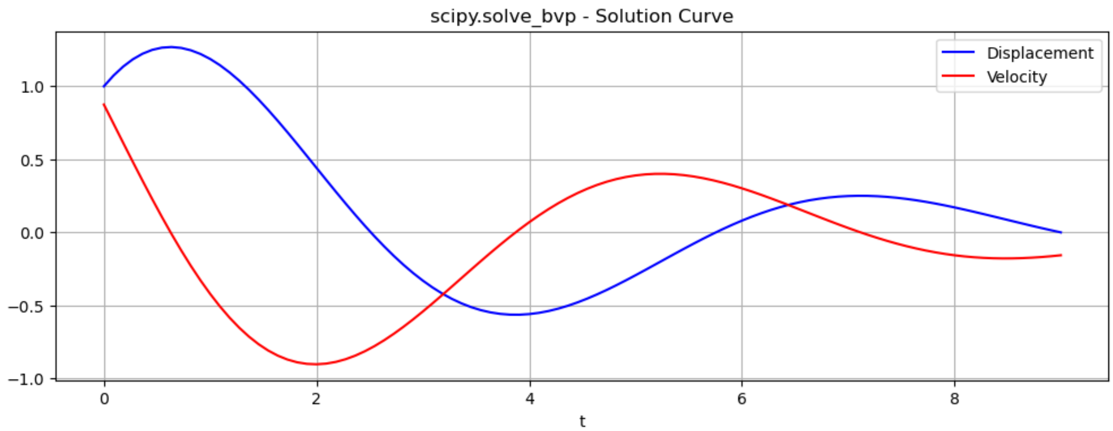
Next, let us extract the data.

[19]:
```
t_bvp = sol.x
x_bvp = sol.y[0]
v_bvp = sol.y[1]
```

We conclude this presentation with the following plot,

[20]:
```
plt.figure(figsize=(12, 4,))
plt.plot(t_bvp, x_bvp, label='Displacement', color='blue')
```

```
plt.plot(t_bvp, v_bvp, label='Velocity', color='red')


plt.xlabel('t')
plt.title('scipy.solve_bvp - Solution Curve')
plt.legend()
plt.grid(True)
plt.show()
```



Sieht gut aus. □