Mathematical Modeling Lab @ Constructor University

Problem 1

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A Prelude

A joint-work between Omar Elshinawy & Mohammad Habibi-Bennani, this project was completed in Spring of 2024 for a class in Mathematical Modeling at Constructor University, under Professor Ivan Ovsyannikov and Mr Dzmitry Rumiantsau.

We start by importing the very basic libraries for this project.

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

Problem 1. Let us consider the equation of a damped harmonic oscillator,

$$\ddot{x} + 2\zeta\dot{x} + x = 0$$

1.0. We start by introducing a meaningful change of variable

$$v = \dot{x},$$

which yields

$$\frac{d}{dt}\, y := \frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{cases} \dot{x} = v \\ \dot{v} = -2\zeta v - x \end{cases}$$

that is a first-order system of equations. We can represent this in python as follows,

[2]:
```
def damped_oscillator(t, y, zeta):
    x, v = y
    dxdt = v
    dvdt = -x - 2 * zeta * v
    return [dxdt, dvdt]
```

with y the vector of position x and velocity v, t the time variable, and ζ the parameter.

It returns the position and velocity derivatives respectively, which respect the aforementioned system.

1.1 Let us solve this system for the parameters

$$\zeta_1 = 0.25, \zeta_2 = 1, \zeta_3 = 2$$

[3]:
```
zeta_1 = 0.25; zeta_2 = 1; zeta_3 = 2
zetas = [zeta_1, zeta_2, zeta_3] # array of parameters
```

and initial conditions

$$x(0) = 1 \;\;\&\;\; v(0) = 0.$$

[4]:
```
x_0 = 1; v_0 = 0
y_0 = [x_0, v_0] # vector of initial conditions
```

Let us set a timeframe of 10 seconds.

$$t \in [0, 10]$$

[5]:
```
t_0 = 0; t_1 = 10
```

1.1.1. Explicit Euler Method

First, observe that our system is given by

$$\frac{d}{dt}y = f(t, y, \zeta) \quad \text{with} \quad y \in \mathbb{R}^2.$$

To approximate a solution to this system, we start by defining the time steps.

$$t_n = t_0 + n \cdot \Delta t \quad \text{for} \quad n = 0, 1, 2, \ldots, N$$

Next, the solution is initialised with the initial condition

$$y_0 = y(t_0).$$

We then iterate over each time step to compute the solution.

$$y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n, \zeta) \quad \text{for} \quad n \in \{0, \ldots, N-1\}$$

In this context: - $ y_n $ is the approximation of $ y(t_n) $ from the previous iterate. - $ f(t_n, y_n, ζ) $ is the value of $ f $ at time $ t_n $, state $ y_n $, and parameter $ ζ$.

The sequence $\{y_n\}_{n\in\mathbb{N}}$ should converge to our solution.

[6]:
```
def explicit_euler(f, y_0, t_0, t_1, dt, zeta):

    # time steps
    t_n = np.arange(t_0, t_1, dt)
    # initialising t x 2 matrix of values with each row denoting (x, v) at time t
```

```
    y_n = np.zeros((len(t_n), len(y_0)))
    # set the initial condition, y(0) := (x_0, v_0) at t=0
    y_n[0] = y_0

    # loop from 1 to N - 1
    for i in range(1, len(t_n)):
        # y_{n+1} = y_n + dt  f(t_n, y_n, \zeta)
        y_n[i] = y_n[i-1] + dt * np.array(f(t_n[i-1], y_n[i-1], zeta))

    # return (x,t) values at time t_n
    return y_n
```

To solve the system, one now simply calls the function with the correct input. We will delay this for one moment.

1.1.2. Implicit Euler Method

The implicit Euler method is unsurprisingly implicit in the following sense.

$$y_{n+1} = y_n + \Delta t \cdot f(t_{n+1}, y_{n+1}, \zeta)$$

We see that y_{n+1} is prevalent in both sides, meaning that for every y_n we have an equation that needs to be solved, of the form

$$y_{n+1} - y_n - \Delta t \cdot f(t_{n+1}, y_{n+1}, \zeta) = 0.$$

Not to distract ourselves from the main objectives, we use `fsolve` to obtain y_{n+1} from the previous equation.

[7]:
```
from scipy.optimize import fsolve
```

Recall that y_n, $\Delta t, \zeta$ are given, so that the equation above indeed yields y_{n+1}, its root.

[8]:
```
def implicit_euler(f, y_0, t_0, t_1, dt, zeta):

    # time steps
    t_values = np.arange(t_0, t_1, dt)
    # initialising t x 2 matrix of values with each row denoting (x, v) at time t
    y_values = np.zeros((len(t_values), len(y_0)))
    # set the initial condition, y(0) := (x_0, v_0) at t=0
    y_values[0] = y_0

    # for every time stamp
    for i in range(1, len(t_values)):
        t = t_values[i]

        # define the implicit function
        def implicit_eq(y_new):
```

```
            return y_new - y_values[i-1] - dt * np.array(f(t, y_new, zeta))

        # solve it using fsolve
        y_values[i] = fsolve(implicit_eq, y_values[i-1])

    # return the array of answers
    return y_values
```

We will also delay calling the function for now. This is for the sake of conciseness.

1.1.3 `scipy.integrate.solve_ivp`

There is no magic here, really.

[9]:
```
from scipy.integrate import solve_ivp
```

We can simply solve the equation for different timesteps and parameters. Usually this is of the form

`solve_ivp(f, [t_0, t_1], y_0, args=(zeta,), t_eval=np.arange(t_0, t_1, dt))`

where `f` would be `damped_oscillator`, `y_0` the initial conditions, `zeta` the argument, and `dt` the preferred timestep.

1.1 Altogether: The Data Dictionary

For conciseness, we will be defining a dictionary `X[dt][zeta]['x' or 'v'][-]` for timestamp dt, parameter \$ζ\$ such that, - `X[-][-][-][0]` is the reference solution using `solve_ivp`; - `X[-][-][-][1]` is the explicit Euler method; - `X[-][-][-][2]` is the implicit Euler method.

We do not go into the details of constructing such a dictionary, as it takes us away from the main goals of this project.

We start by defining `tstamps`, an array of timestamps $\Delta t \in [0.01, 1]$.

[10]:
```
tstamps = np.linspace(0.01, 1, 100)
#print(tstamps)
```

From here we can choose two timesteps of different size. This will turn out to be a useful generalisation for the sake of this discussion.

[11]:
```
dt_0 = tstamps[0]; dt_1 = tstamps[14]
```

This is just

$$\Delta t_0 = 0.01 \ \& \ \Delta t_1 = 0.15,$$

which we call directly from the `tstamps` array to avoid numerical rounding errors.

[12]:
```
print(f"dt_0 = {dt_0} | dt_1 = {dt_1:.2f}")
```

```
dt_0 = 0.01 | dt_1 = 0.15
```

[13]:
```
X = {}

# for all timestamps
for dt in tstamps:
    X[dt] = {} # Initialize dictionary

    # for all zetas
    for zeta in zetas:
        X[dt][zeta] = {'x': [], 'v': []}  # Initialize placeholders x, v

        # More data handling
        X[dt][zeta]['x'] = [None, None, None]
        X[dt][zeta]['v'] = [None, None, None]

        # Reference Solution solve_ivp
        reference = solve_ivp(damped_oscillator, [t_0, t_1], y_0, args=(zeta,),␣
 ↪t_eval=np.arange(t_0, t_1, dt))
        X[dt][zeta]['x'][0] = reference.y[0] # y[0] gives x
        X[dt][zeta]['v'][0] = reference.y[1] # y[1] gives v

        # Explicit Euler Method
        explicit = explicit_euler(damped_oscillator, y_0, t_0, t_1, dt, zeta)
        X[dt][zeta]['x'][1] = explicit[:, 0] # 0 gives x
        X[dt][zeta]['v'][1] = explicit[:, 1] # 1 gives v

        # Implicit Euler Method
        implicit = implicit_euler(damped_oscillator, y_0, t_0, t_1, dt, zeta)
        X[dt][zeta]['x'][2] = implicit[:, 0] # 0 gives x
        X[dt][zeta]['v'][2] = implicit[:, 1] # 1 gives v
```

And now, finding the right function has never been so simple. One only needs to call the correct function (with the correct parameters) to solve the system.

1.2. The Phase Space Trajectory.

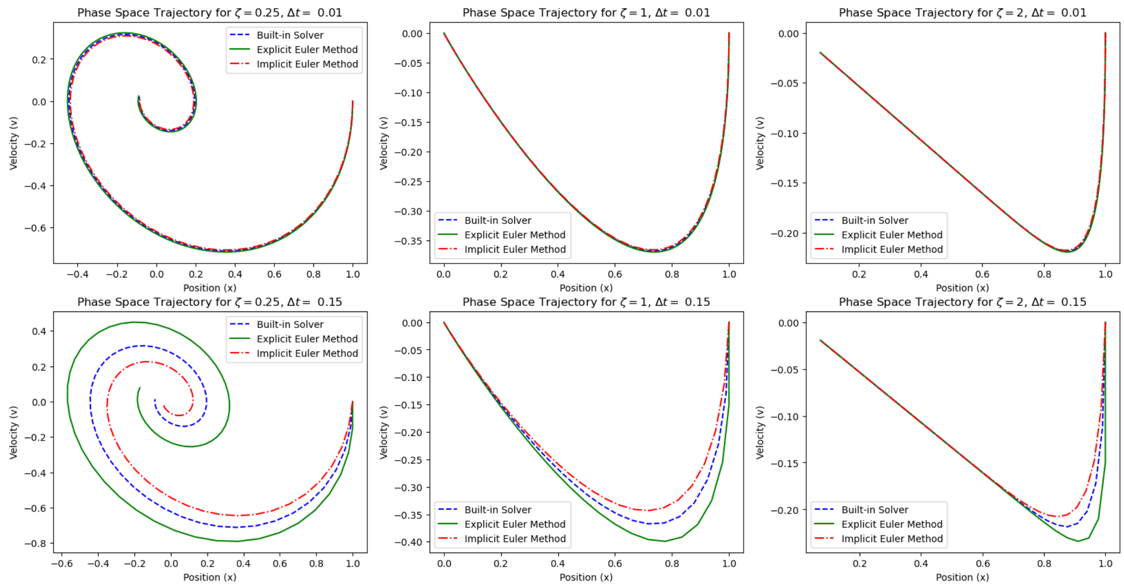
The phase space trajectory is a plot of displacement against velocity. We proceed to demonstrate that below for

$$\Delta t_0 = 0.01 \ \& \ \Delta t_1 = 0.15$$

as well as for $\zeta \in \{0.25, 1, 2\}$.

[14]:
```
fig, axs = plt.subplots(2, 3, figsize=(20, 10))  # Figure with 3 subplots␣
 ↪horizontally arranged

# index 1
i = 0;
```

```
# For big and small timestamps
for dt in [dt_0, dt_1]:

    # index 2
    j = 0;

    # Method labels
    methods = ["Built-in Solver", "Explicit Euler Method", "Implicit Euler␣
↪Method"]

    # Loop over each zeta value
    for zeta in zetas:

        # solve_ivp ; X[-][-][-][0]

        axs[i,j].plot(X[dt][zeta]['x'][0], X[dt][zeta]['v'][0],
                        label=methods[0], linestyle='--', color='blue')  # Plot␣
↪built-in solver method

        # Explicit Euler ; X[-][-][-][1]

        axs[i,j].plot(X[dt][zeta]['x'][1], X[dt][zeta]['v'][1],
                        label=methods[1], linestyle='-', color='green')  # Plot␣
↪explicit method

        # Implicit Euler ; X[-][-][-][2]

        axs[i,j].plot(X[dt][zeta]['x'][2], X[dt][zeta]['v'][2],
                        label=methods[2], linestyle='-.', color='red')  # Plot␣
↪implicit method

        axs[i,j].set_xlabel('Position (x)')
        axs[i,j].set_ylabel('Velocity (v)')
        axs[i,j].set_title(f"Phase Space Trajectory for $\zeta = {zeta}$,␣
↪$\Delta t = $ {dt:.2f}")
        axs[i,j].legend()

        # increments twice
        j += 1
    # increments once
    i += 1
```



One can of course. introduce more coefficients ζ or manipulate the choices of $\Delta t_0, \Delta t_1$ when convenient.

1.3. Errors

Since we already have the built-in `scipy.integrate.solve_ivp`, it will be our reference when checking the accuracy of our two Euler schemes. To make nice log-log graphs, we simply summon the `plt.loglog()` command. In order to plot the error versus step size, we would like to

a) call timestamps Δt between $[0.01, 1]$;
b) find the error ϵ for each timestamp Δt, which will be the average error in comparison to `solve_ivp`;
c) plot the results in the form $(\Delta t, \epsilon)$.

a) The variable `tstamps` has already been introduced with values $\Delta t_i \in [0.1, 1]$.

b) Now that we have the timestamps, we can start computing the errors. We will introduce the following convention:

- `E[-][-][-][1]` is the explicit Euler ***error*** for (x or v);
- `E[-][-][-][2]` is the implicit Euler ***error*** for (x or v);

in harmony with - `X[-][-][-][1]` $\mapsto$ `explicit_euler`; - `X[-][-][-][2]` $\mapsto$ `implicit_euler`.

[15]:
```python
E = {}

for dt in tstamps:
    E[dt] = {} # Initialize dictionary
    for zeta in zetas:
        E[dt][zeta] = {'x': [], 'v': []}  # Initialize placeholders x, v
```

```
        # More data handling
        E[dt][zeta]['x'] = [None, None, None]
        E[dt][zeta]['v'] = [None, None, None]

        # displacement error
        E[dt][zeta]['x'][1] = abs(X[dt][zeta]['x'][0] - X[dt][zeta]['x'][1]) #␣
↪|reference - explicit|
        E[dt][zeta]['x'][2] = abs(X[dt][zeta]['x'][0] - X[dt][zeta]['x'][2]) #␣
↪|reference - implicit|

        # velocity error
        E[dt][zeta]['v'][1] = abs(X[dt][zeta]['v'][0] - X[dt][zeta]['v'][1]) #␣
↪|reference - explicit|
        E[dt][zeta]['v'][2] = abs(X[dt][zeta]['v'][0] - X[dt][zeta]['v'][2]) #␣
↪|reference - implicit|
```

Notice now that for instance, the error `E[0.01][zeta]['x or v'][1 or 2]` has 1000 elements.

[16]:
```
len(E[0.01][0.25]['x'][1])
```

[16]: 1000

It only makes sense to take the average of these values for each timestamp `dt` so that we can get the full picture.

[17]:
```
for dt in tstamps:
    for zeta in zetas:
        E[dt][zeta]['x'][1] = np.mean(E[dt][zeta]['x'][1])
        E[dt][zeta]['v'][1] = np.mean(E[dt][zeta]['v'][1])

        E[dt][zeta]['x'][2] = np.mean(E[dt][zeta]['x'][2])
        E[dt][zeta]['v'][2] = np.mean(E[dt][zeta]['v'][2])
```

With that off the checklist, we are ready to look at some nice graphs.

c) Choose your preferred value for the parameter $\zeta \in \{0.25, 1, 2\}$. We note that the dictionary `X` is only defined for these values, but that can be easily changed by appending the desired ζ to `zetas`.

[18]:
```
zeta = 2
```

Then, collect the error averages for each method and variable. These will be the arrays of y-values in the $(\Delta t, \epsilon)$ construction.

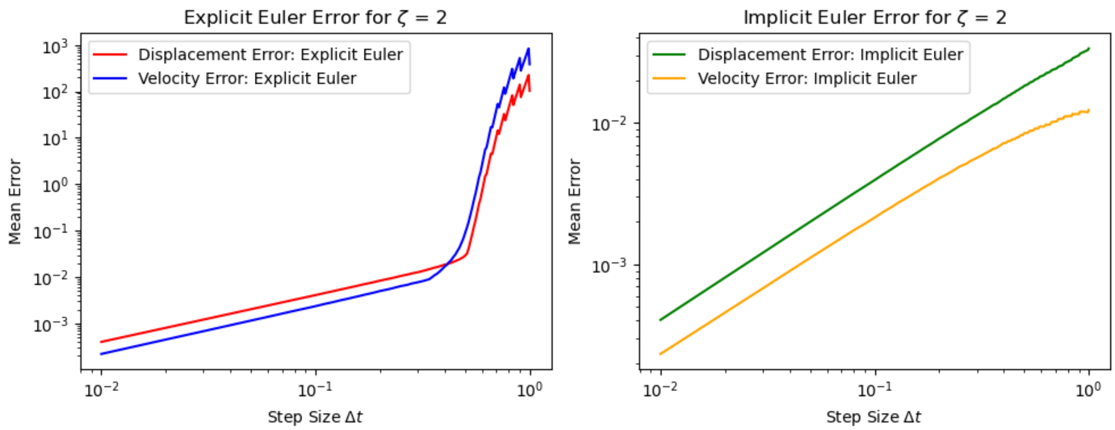
[19]:
```
i=0; tstmps = len(tstamps)

# initialization
explicit_error_x = np.zeros(tstmps)
implicit_error_x = np.zeros(tstmps)
explicit_error_v = np.zeros(tstmps)
implicit_error_v = np.zeros(tstmps)
```

```python
# for each timestep
for dt in tstamps:

    # explicit error averages
    explicit_error_x[i] = E[dt][zeta]['x'][1]
    explicit_error_v[i] = E[dt][zeta]['v'][1]
    # implicit error averages
    implicit_error_x[i] = E[dt][zeta]['x'][2]
    implicit_error_v[i] = E[dt][zeta]['v'][2]

    # increment
    i += 1
```

Next, one simply plots timestamps Δt versus error ϵ for each scheme.

[20]:
```python
# Plot setup
fig, axs = plt.subplots(1, 2, figsize=(10, 4))

# Displacement Error for Explicit Euler
axs[0].loglog(tstamps, explicit_error_x,
                 label="Displacement Error: Explicit Euler", linestyle='-',␣
 ↪color='red')

# Velocity Error for Explicit Euler
axs[0].loglog(tstamps, explicit_error_v,
                 label="Velocity Error: Explicit Euler", linestyle='-',␣
 ↪color='blue')

axs[0].set_title(f'Explicit Euler Error for $\zeta$ = {zeta}')
axs[0].set_xlabel('Step Size $\Delta t$')
axs[0].set_ylabel('Mean Error')
axs[0].legend()

# Displacement Error for Implicit Euler
axs[1].loglog(tstamps, implicit_error_x,
                 label="Displacement Error: Implicit Euler", linestyle='-',␣
 ↪color='green')

# Velocity Error for Implicit Euler
axs[1].loglog(tstamps, implicit_error_v,
                 label="Velocity Error: Implicit Euler", linestyle='-',␣
 ↪color='orange')

# Some details
axs[1].set_title(f'Implicit Euler Error for $\zeta$ = {zeta}')
axs[1].set_xlabel('Step Size $\Delta t$')
```

```python
axs[1].set_ylabel('Mean Error')
axs[1].legend()


# Adjust layout and display plot
plt.tight_layout()
plt.show()
```



We make the following conclusions:

- the **Explicit Euler Method** has an error of order $\mathcal{O}(\Delta t^2)$;
- the **Implicit Euler Method** similarly yields an error of order $\mathcal{O}(\Delta t^2)$.

This conlcudes the exercise. □