Fibrations & Applications to Computations of Higher Homotopy Groups

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Algebraic Topology
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Overview

Main Results

Theorem (The Path-loop Fibration)

$\pi_n(\Omega X) \cong \pi_{n+1}(X)$

Theorem (The Hopf Fibration)

$\pi_n(S^3) \cong \pi_n(S^2)$ *for* $n \geq 3$

The Hopf Fibration

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$\pi_n(S^3) \cong \pi_n(S^2)$ *for* $n \geq 3$

1. Fibrations induce Long Exact Sequences

Theorem

Let $p : E \to_p B$ be a Serre fibration. Choose $b_0 \in B$ and $x_0 \in F$ such that $p^{-1}(b_0) = x_0$. Then, there exists a map

$$p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$$

that is an isomorphism for all $n \geq 1$.

Corollary

B path-connected $\implies$ Long Exact Sequence

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \dots$$

2. Higher Homotopy Groups of S^1

Proposition

$$\pi_n(S^1) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}$$

3. Lemma on Exact Sequences

Definition (Exact Sequence)

A family of groups $\{G_i\}_{i=1}^{n}$, a family of homomorphisms $\{f_i\}_{i=1}^{n-1}$

$$G_1 \rightarrow_{f_1} G_2 \rightarrow_{f_2} \cdots \rightarrow_{f_{n-1}} G_n$$

Sequence is exact $\implies$ for all $i \in \{1, \ldots, n-1\}$,

$$\mathrm{im}(f_i) = \ker(f_{i+1})$$

Lemma

$$0 \rightarrow_{f_1} G_1 \rightarrow_{f_2} G_2 \rightarrow_{f_3} 0 \textit{ is exact} \iff G_1 \cong G_2.$$

First Main Result

Theorem

$\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$

Proof. This is a Fibration

$$S^1 \hookrightarrow S^3 \rightarrow_\eta S^2$$

1. Fibrations induce exact sequences

$$\cdots \rightarrow \pi_n(S^1) \rightarrow \pi_n(S^3) \rightarrow \pi_n(S^2) \rightarrow \pi_{n-1}(S^1) \rightarrow \ldots \quad .$$

2. $\pi_n(S^1) = 0$ for $n \geq 2$

$$\cdots \rightarrow 0 \rightarrow \pi_n(S^3) \rightarrow \pi_n(S^2) \rightarrow 0 \rightarrow \ldots$$

3. Isomorphism of groups ∎

The Path-loop Fibration

Theorem (The Path-loop Fibration)

$\pi_n(\Omega X) \cong \pi_{n+1}(X)$

1. Fibrations induce Long Exact Sequences

Theorem

Let $p : E \to_p B$ be a Serre fibration. Choose $b_0 \in B$ and $x_0 \in F$ such that $p^{-1}(b_0) = x_0$. Then, there exists a map

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Corollary

B path-connected $\implies$ Long Exact Sequence

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \dots$$

2. The Path Space is Contractible

Definition

(Homotopy of Spaces) $X \simeq Y$ if there exists

$$f : X \to Y \;\;\&\;\; g : Y \to X$$

such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. A **contractible** space is homotopy equivalent to a point.

Lemma

PX is contractible.

Lemma

PX is contractible $\iff \pi_n(PX) = 0$

3. Lemma on Exact Sequences

Definition (Exact Sequence)

A family of groups $\{G_i\}_{i=1}^{n}$, a family of homomorphisms $\{f_i\}_{i=1}^{n-1}$

$$G_1 \to_{f_1} G_2 \to_{f_2} \cdots \to_{f_{n-1}} G_n$$

Sequence is exact $\implies$ for all $i \in \{1, \dots, n-1\}$,

$$\mathrm{im}(f_i) = \ker(f_{i+1})$$

Lemma

$$0 \to_{f_1} G_1 \to_{f_2} G_2 \to_{f_3} 0 \textit{ is exact } \iff G_1 \cong G_2.$$

Second Main Result

Theorem

$\pi_n(\Omega X) \cong \pi_{n+1}(X)$

Proof. This is a Fibration

$$\Omega_b X \hookrightarrow P(X,b) \to X_b$$

1. Fibrations induce exact sequences

$$\cdots \to \pi_{n+1}\big(P(X,b)\big) \to \pi_{n+1}(X) \to \pi_n(\Omega X) \to \pi_n\big(P(X,b)\big) \to \ldots \quad .$$

2. PX contractible $\iff \pi_n(PX) = 0$

$$\cdots \to 0 \to \pi_{n+1}(X) \to \pi_n(\Omega X) \to 0 \to \ldots \quad .$$

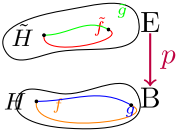
3. Isomorphism of groups □

The Lifting Problem

Definition

$p : E \to B$ has the homotopy lifting property wrt X if

1. for all maps $f : X \to E$



2. for all homotopies $G : X \times [0,1] \to B$ of the map $p \circ f$

there exists a homotopy $F : X \times [0,1] \to E$ such that

a) $F_0 = f$ *the homotopy F starts with f*

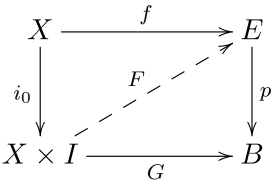
b) $p \circ F = G$. *the homotopy on E projects to the homotopy on B*

Fibration

Definition (Fibration)

A (Hurewicz) fibration is a surjection $p : E \to B$ that satisfies the homotopy lifting property for all spaces.

Commutative Diagram



Relative HLP

Definition

Let

$$T := (X \times \{0\}) \cup (A \times [0,1]).$$

We say p has the homotopy lifting property for a pair (X, A) if

1. for any homotopy $F : X \times [0,1] \to B$
2. for any lifting $\tilde{f} : T \to E$ of $f = F\big|_T$

there exists a homotopy $G : X \times [0,1] \to E$ such that $p \circ G = F$ and $\tilde{F}\big|_T = \tilde{g}$.

Remark

Notice that $A = \phi \implies T = X \times \{0\}$ gives the standard definition of HLT.

p_* is Surjective

Theorem

Let $p : E \to_p B$ be a Serre fibration. Choose $b_0 \in B$ and $x_0 \in F$ such that $p^{-1}(b_0) = x_0$. Then, there exists a map

$$p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$$

that is an isomorphism for all $n \geq 1$.

Proof. **1. p_* is surjective**

p_* is Injective

Theorem

Let $p : E \to_p B$ be a Serre fibration. Choose $b_0 \in B$ and $x_0 \in F$ such that $p^{-1}(b_0) = x_0$. Then, there exists a map

$$p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$$

that is an isomorphism for all $n \geq 1$.

Proof. **2. p_* is injective**

Key Corollary

Corollary

If B is path-connected. Then there is an induced long exact sequence on homotopy groups, given by

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \dots \quad .$$

Statement without proof. The isomorphism induces the exact sequence for the pair (E, F);

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(E, F)$$

Proof.

□

Thanks!

Questions?