Fibrations & Applications to Computations of Higher Homotopy Groups

Omar Elshinawy

Spring 2024

Abstract

This is an expository paper on fibrations in the context of higher homotopy groups. The paper does not assume any pre-requisite knowledge, adopting a rather axiomatic approach. We direct our focus towards fibrations and show how one can apply this powerful machinery to compute homotopy groups.

One main result regards the loop-space fibration, which shifts the homotopy group by one degree. Another highly non-trivial result utilisies the Hopf fibration to prove that the homotopy groups of S^2 and S^3 coincide for $n \geq 3$.

The goal of this paper is to trivialise and familiarise, and to equip the reader with the right ideas for a more developed treatment of the subject.

Contents

0 Algebraic Preliminaries 2

1 The Simplest Homotopy Group 3

2 More Complicated Homotopy Groups 7

3 A Glimpse into $\pi_n(S^k)$ 10

4 Fibrations & The Lifting Problem 12

5 Applications to Homotopy Groups 17

Acknowledgements 19

References 19

0 Algebraic Preliminaries

We start with some algebraic preliminaries.

Definition 0.1 (Group)**.** Let G be an arbitrary set, endowed with a binary operation $*$. We say $(G, *)$ is a group if for all $g_1, g_2, g_3 \in G$,

1. $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$ — *$*$ is associative*
2. $\exists e \in G \; : \; e \cdot g_1 = g_1 \cdot e = g_1$ — *there exists an identity element*
3. $\exists \bar{g_1} \in G \; : \; g_1 * \bar{g_1} = \bar{g_1} * g_1 = e$ — *as well as an inverse for every element*

Definition 0.2 (Group Homomorphism)**.** Let $(G, *)$ and $(H, \cdot)$ be two groups. We say a function $h : G \to H$ induces a homomorphism if for all $u, v \in G$, we have

$$h(u * v) = h(u) \cdot h(v).$$

Corollary 0.3. *h sends e_G, the identity on G, to e_H, the identity on h.*

Proof. We know that $\mathrm{im}(h) \in H$, so we apply the left inverse of $h(e_G)$, that is $\bar{h}(e_G)$.

$$e_H =: \bar{h}(e_G) \cdot h(e_G * e_G) =_{\mathbf{0.2}} \bar{h}(e_G) \cdot h(e_G) \cdot h(e_G) := h(e_G) \quad \square$$

Note that h need not be bijective, so we emphasize that with the next definition.

Definition 0.4 (Group Isomorphism)**.** If h is bijective, then it induces an isomorphism. We write $G \cong H$ and sometimes $G = H$.

Definition 0.5 (Long Exact Sequence)**.** Let $\{G_i\}_{i=1}^n$ be a family of groups. A sequence

$$G_1 \to_{f_1} G_2 \to_{f_2} \cdots \to_{f_{n-1}} G_n$$

induced by **homomorphisms** $\{f_i\}_{i=1}^{n-1}$ is said to be long-exact if for all $i \in \{1, \dots, n-1\}$,

$$\mathrm{im}(f_i) = \ker(f_{i+1})$$

Remark. A short exact sequence is the special case where $G_1 = G_n = 0.$[1]

Not only is the following lemma a great example that grasps the idea of an exact sequence; it also plays a key role in our discussions of homotopy groups.

Lemma 0.6. *Let G_1, G_2 be two groups. Then,*

$$0 \to_{f_1} G_1 \to_{f_2} G_2 \to_{f_3} 0 \textit{ is exact} \iff G_1 \cong^{2} G_2.$$

Proof. f_i are homomorphisms, therefore f_i sends $0 \mapsto 0$ (0.3). The isomorphism yields

$$\begin{aligned} G_1 \cong G_2 &\iff \mathrm{im}(f_2) = G_2 \wedge \ker(f_2) = 0 \\ &\iff \mathrm{im}(f_1) := 0 =: \ker(f_2) \wedge \; \mathrm{im}(f_2) := G_2 =:\ker(f_3) \end{aligned} \quad \square$$

[1] From now on, 0 is thought of as the group identity.
[2] $G_1 \cong G_2 \iff G_1$ is isomorphic to G_2

1 The Simplest Homotopy Group

The idea behind $\pi_1(X)$ is rougly summarized in "fitting loops into space," with the key observation that a loop can be deformed into a circle. Looking at these groups can reveal useful info about the structure of a space. The fundamental group is also an algebraic invariant, meaning that two spaces X, Y with different fundamental groups i.e. $(X \ncong Y)$ are two different spaces which are not homotopic, so that is already useful.

The concept of higher homotopy groups is a generalization of the fundamental idea of fitting loops into space. Thus, it only makes sense to first intuit this idea as we explore more complicated constructions.

Definition 1.1 (Loop)**.** A loop $\gamma : [0, 1] \to X$ is a continuous path with the property that

$$\gamma(0) = \gamma(1) = x_0 \in X$$

i.e. it starts and ends at the same point.

Definition 1.2 (Homotopy of Loops)**.** Let $\gamma,\ \gamma' : [0, 1] \to X$ be two loops such that for some $x_0 \in X$ we have

$$\gamma(0) = \gamma(1) = x_0 = \gamma'(0) = \gamma'(1).$$

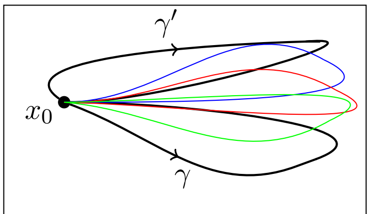
We say $h : [0, 1] \times [0, 1] \to X$ is a homotopy between γ, γ' if for all $s, t \in [0, 1]$,

1. $h(0, t) = \gamma(t)$ — *homotopy starts with the loop γ*
2. $h(1, t) = \gamma'(t)$ — *homotopy ends with the loop γ'*
3. $h(s, 0) = h(s, 1) = x_0$ — *base point x_0 is fixed at the endpoints regardless of s*

Finally, we demand h to be continuous.

Remark. A homotopy of paths holds for $h(s, 0),\ h(s, 1)$ yielding both ends of the path.

It is often useful to think of the homotopy as the continuous deformation of loops in between γ and γ', with fixed basepoint x_0. This is best illustrated below.



Homotopy Class of γ, γ'

Corollary 1.3. *The homotopy h defines an equivalence relation $\simeq$ over all paths in X.*

Proof.

1. **Reflexivity.** It is easy to see that $\gamma \simeq \gamma$ with the trivial homotopy $h(s, t) = \gamma(t)$ for all $s \in [0, 1]$.

2. **Symmetry.** Let $\gamma \simeq \gamma'$. Then, there exists continuous $h(s,t)$ such that

   - $h(0,t) = \gamma(t)$;
   - $h(1,t) = \gamma'(t)$;
   - $h(s,0) = h(s,1) = x_0$.

   Next, define $\bar{h}(s,t) = h(1-s,t)$. Clearly $\bar{h}$ is inherently continuous, and

   - $h(0,t) = \gamma(t) =: \bar{h}(1,t)$;
   - $h(1,t) = \gamma'(t) =: \bar{h}(0,t)$;
   - $h(s,0) = \bar{h}(s,0) = x_1 = h(s,1) = \bar{h}(s,1)$. *$h$ fixes x_0 regardless of s*

   Therefore $\gamma' \simeq \gamma$.

3. **Transitivity.** Let $\gamma_1 \simeq \gamma_2$ and $\gamma_2 \simeq \gamma_3$. Then, define

$$H(s,t) = \begin{cases} h_1(2s,t) & 0 \le s \le \frac{1}{2} \\ h_2(2s-1,t) & \frac{1}{2} \le s \le 1 \end{cases}$$

   with h_1 a homotopy of γ_1, γ_2 and h_2 a homotopy of γ_2, γ_3.

   - $H(0,t) = h_1(0,t) := \gamma_1(t)$;
   - $H(1,t) = h_2(1,t) := \gamma_3(t)$;
   - $H(s,0) = \begin{cases} h_1(s,0) := x_0 & 0 \le s \le \frac{1}{2} \\ h_2(2s-1,0) := x_0 & \frac{1}{2} \le s \le 1 \end{cases} = x_0$;
   - $H(s,1) = \begin{cases} h_1(s,1) := x_0 & 0 \le s \le \frac{1}{2} \\ h_2(2s-1,1) := x_0 & \frac{1}{2} \le s \le 1 \end{cases} = x_0$.

   Therefore by the homotopy H we have that $\gamma_1 \simeq \gamma_3$.

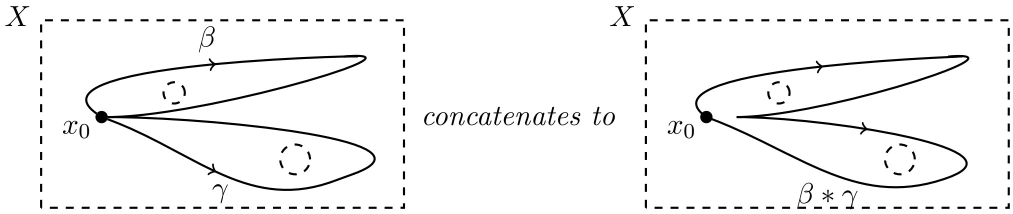
A lengthy yet easy argument. □

Definition 1.4 (Concatenation of Loops.)**.** Let γ, β be two loops in X such that $\gamma(0)$ coincides with $\beta(1)$. We define

$$(\gamma * \beta)(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

to be the concatenation of γ, β.

Example 1.4. An interesting observation here is that concatenation yields a new homotopy class different than the other classes. Indeed, $[\gamma] \neq [\beta] \neq [\gamma * \beta.]$



The concatenation operation induces a group structure on the following set.

Definition 1.5 (The Fundamental Group)**.** Let X be a topological space with a base point x_0. We define

$$\pi_1(X) = \{\gamma : [0,1] \to X \mid \gamma \text{ continuous, } \gamma(0) = \gamma(1) = x_0\}/\simeq$$

to be the set of equivalence classes of loops under homotopy.

Theorem 1.6. *The set $\pi_1(X, x_0)$ endowed with concatenation $*$ is a group.*

Proof. Recall that homotopy fixes endpoints, so we can rest assured that the operation $*$ is well defined. We proceed with the standard group axioms.

1. **Associativity.** For homotopy classes of loops $[\gamma], [\beta], [\alpha]$ we have

$$\gamma * (\beta * \alpha) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \beta(4t-2) & \frac{1}{2} \le t \le \frac{3}{4} \\ \alpha(4t-3) & \frac{3}{4} \le t \le 1 \end{cases} \qquad \& \quad (\gamma * \beta) * \alpha = \begin{cases} \gamma(4t) & 0 \le t \le \frac{1}{4} \\ \beta(4t-2) & \frac{1}{4} \le t \le \frac{1}{2} \\ \alpha(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

the concatenation of loops. It suffices to show that $(\gamma * \beta) * \alpha \simeq \gamma * (\beta * \alpha)$. Choose

$$h(s,t) = \begin{cases} \gamma\big(2t \cdot s + 4t \cdot (1-s)\big) & 0 \le t \le (\frac{s}{2} + \frac{1-s}{4}) \\ \beta\big(4t-2\big) & \frac{s}{2} + \frac{1-s}{4} \le t \le \frac{3s}{4} + \frac{1-t}{2} \\ \alpha\big((4t-3) \cdot s + (2t-1) \cdot (1-s)\big) & \frac{3s}{4} + \frac{1-s}{2} \le t \le 1 \end{cases}$$

and then we have that

(a) $h(0,t) = \begin{cases} \gamma\big(4t\big) & 0 \le t \le \frac{1}{4} \\ \beta\big(4t-2\big) & \frac{1}{4} \le t \le \frac{1}{2} \\ \alpha\big(2t-1\big) & \frac{1}{2} \le t \le 1 \end{cases} =: (\gamma * \beta) * \alpha$ *start of homotopy*

(b) $h(1,t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \beta(4t-2) & \frac{1}{2} \le t \le \frac{3}{4} \\ \alpha(4t-3) & \frac{3}{4} \le t \le 1 \end{cases} =: \gamma * (\beta * \alpha)$ *end of homotopy*

(c) $h(s,0) = \gamma\big(0\big)$ *base point of $(\gamma * \beta) * \alpha$ is independent of s*

(d) $h(s,1) = \alpha\big(s + (1-s)\big) = \alpha(1)$ *base point of $\gamma * (\beta * \alpha)$ is independent of s*

and one only needs to check that h is continuous at $(\frac{t}{2}+\frac{1-t}{4})$ and $(\frac{3t}{4}+\frac{1-t}{2})$ to see the homotopy.

2. **Identity Element.** Let $id_{x_0}(t)$ be the constant loop x_0 for all t. Then clearly $id_{x_0}(0) = id_{x_0}(1) = x_0$ and therefore $[id_{x_0}] \in \pi_1(X)$. Next, choose a γ and consider

$$\gamma * id_{x_0} = \begin{cases} \gamma\big(2t\big) & 0 \le t \le \frac{1}{2} \\ id_{x_0} = \gamma(1) & \frac{1}{2} \le t \le 1 \end{cases} \quad \& \quad id_{x_0} * \gamma = \begin{cases} id_{x_0} = \gamma(0) & 0 \le t \le \frac{1}{2} \\ \gamma\big(2t-1\big) & \frac{1}{2} \le t \le 1 \end{cases} .$$

We get that $\gamma \simeq (\gamma * id_{x_0})$ and $\gamma \simeq (id_{x_0} * \gamma)$ as they are only reparameterizations of each other. Hence, the equivalence classes

$$[\gamma * id_{x_0}] = [id_{x_0} * \gamma] = [\gamma]$$

are all the same.

3. **Inverse.** Choose $\gamma \in [\gamma]$, and let $\bar{\gamma}(t) = \gamma(1-t)$, and let $\gamma(0) = x_0$. We show that $id_{x_0} \simeq \gamma * \bar{\gamma}$ which in turn implies $[\gamma] * [\bar{\gamma}] = [id_{x_0}]$. For

$$(\gamma * \bar{\gamma})(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \bar{\gamma}(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} ,$$

define the homotopy

$$h(s,t) = \gamma(st) * \bar{\gamma}(st) := \begin{cases} \gamma(2st) & 0 \le t \le \frac{1}{2} \\ \bar{\gamma}(2st-1) & \frac{1}{2} \le t \le 1 \end{cases} .$$

We get that

(a) $h(0,t) = \begin{cases} \gamma(0) := x_0 & 0 \le t \le \frac{1}{2} \\ \bar{\gamma}(1) := \gamma(0) & \frac{1}{2} \le t \le 1 \end{cases} =: id_{x_0}$

(b) $h(1,t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \bar{\gamma}(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} =: \gamma * \bar{\gamma}$

(c) $h(s,0) = \gamma(0) := x_0 =: \bar{\gamma}(1) = h(s,1)$ *the base point is independent of s*

and therefore $\gamma * \bar{\gamma} \simeq id_{x_0} \implies [\gamma] * [\bar{\gamma}] = [id_{x_0}]$. Using the homotopy

$$\bar{h} = \bar{\gamma}(st) * \gamma(st)$$

one can show that indeed $[\bar{\gamma}] * [\gamma] = [id_{x_0}]$.

The claim follows immediately. □

The problem, however, with $\pi_1(X)$ is that it ends at the two skeleton. Beyond that, it cannot consistently reveal info about higher-dimensional spaces. A natural idea is to generalise this notion by fitting higher-dimensional spheres into X, thus revealing more information. This is precisely the idea behind π_n, that is fitting $n-$spheres into space. In the following section, we aim our attention at formalizing this idea.

2 More Complicated Homotopy Groups

We begin by introducing some new notation.

Definition 2.1 ($n-$Dimensional Cube & Boundary)**.** Let $I = [0, 1]$ be the unit interval. We define

$$I^n := [0, 1]^n$$

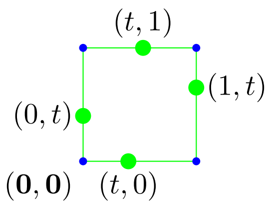
to be the $n-$dimensional cube. We denote its boundary by ∂I^n, given by

$$\partial I^n := \{(t_1, \dots, t_n) \in I^n \mid \text{there exists a } t_i \in \{0, 1\}\}$$

with the convetion $\partial I^0 = \{\}$.

Example 2.1. The boundary of the two dimensional unit cube is given by its four sides.

$$\partial I^2 := \{(0, t) \cup (1, t) \cup (t, 0) \cup (t, 1) \mid t \in [0, 1]\}.$$



It is perhaps now easier to see now why any point with 0 or 1 in its co-ordinates is automatically on the boundary. Next, let us see how we can naturally extend the notion of a one-dimensional loop.

Definition 2.2 (n-Sphere Maps)**.** Let I^n be the $n-$dimensional unit cube. On a pointed topological space (X, x_0), we define

$$f : I^n \to X$$

to be an $n-$sphere map f, if the boundary is mapped to some point $x_0 \in X$

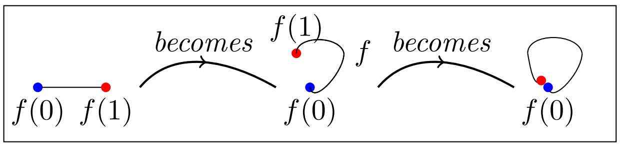
$$f(\partial I^n) = x_0.$$

Equivalently, one can define the $n-$sphere map as follows.

$$f : S^n \to X \quad \textit{for} \quad S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}.$$

Remark. The map f is indeed homeomorphic to S^n.

Example 2.2 (Mapping the Boundary)**.** We re-visit the classical case with $n = 1$ where



so we get that mapping the $1-$dimensional boundary to one point gives us a circle S^1!

For $n = 2$, grab a piece of paper, and hold it by its four ends. If you extend the rest of its boundary to this point, you get something that looks like a sphere S^2.

Indeed, the previous example delivers the right intuition across as to why we map the boundary to a point. In full generality, mapping ∂I^n to a base point x_0 gives an $n-$sphere S^n based at x_0. Let us now extend the notion of homotopy in a similar fashion.

Definition 2.3 (Generalized Homotopy of Maps)**.** Let f, g be two $n-$sphere mappings into X. We say f, g are homotopic if there exists a homotopy

$$H : I \times \partial I^n \to X$$

such that for all $t \in \partial I^n$ and all $s \in I$,

1. $H(0, t) = f$ *the homotopy starts with* f
2. $H(1, t) = g$ *the homotopy ends with* g
3. $H(s, \partial I^n) = x_0$ *the image of the boundary is* x_0, *independent of* s

with H continuous.

We can already see that the definition practically generalises **Definition 1.2**. Here we have a homotopy of spheres, for which the boundary is mapped to the same x_0. In the case of a loop, the boundary ∂I consists of just $\{0, 1\}$ which are mapped to x_0.

Corollary 2.4. *$H : I \times \partial I^n \to X$ is an equivalence relation over n-sphere maps in X.*

Proof. Follows immediately from **Corollary 1.3** with the minor tweak that t in $H(s, t)$ is thought of as an element of ∂I^n. □

Definition 2.5 (Higher Homotopy Groups)**.** Let (X, x_0) be a pointed topological space. We define

$$\pi_n(X, x_0) := \{f : [0, 1]^n \to X \mid f(\partial I^n) = x_0\}/\simeq$$

to be the **n−th homotopy group**, that is the set of equivalence classes of $n-$sphere maps with base point $x_0 \in X$.

Remark. For $\pi_0(X, x_0)$ we know that $I^0 := \{0, 1\}$. Since δI^0 is empty, we map $1 \mapsto x_0$ and consider $f(0) \in X$. If $f, \bar{f}$ are homotopic then we find a path that connects $f(0)$ with $\bar{f}(0)$. Therefore we say that $\pi_0(X, x_0)$ is the set of equivalence classes of path-connected components of X.

Definition 2.6 (Concatenation of $n-$Spheres)**.** For $f, g \in \pi_n(X)$, we define $f * g$ by

$$(f * g)(t_1, \ldots, t_n) := \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \ldots, t_n) & \frac{1}{2} \leq t_1 \leq 1 \end{cases} .$$

Remark. We immediately observe that it is equivalent to **Definition 1.4** for concatenation of loops, applied to the first co-ordinate t_1. One could also define

$$(f *_i g)(t_1, \dots, t_n) \begin{cases} f(t_1, t_2, \dots, 2t_i, \dots, t_n) & 0 \leq t_i \leq \frac{1}{2} \\ g(t_1, t_2, \dots, 2t_i - 1, \dots, t_n) & \frac{1}{2} \leq t_i \leq 1 \end{cases} .$$

Interestingly enough, there is an explicit homotopy equivalence $f * g \simeq f *_i g$,

$$H(s,t) := s \cdot (f *_i g)(t) + (1-s) \cdot (f * g)(t)$$

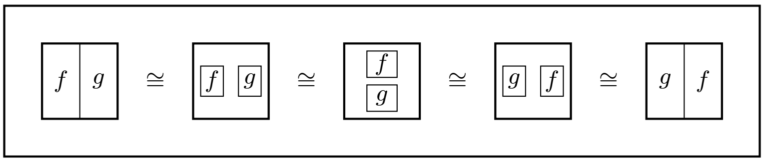
which is sufficient for π_n where we consider equivalence classes of homotopies.

Corollary 2.7. *The sets $\pi_n(X, x_0)$ endowed with concatenation $*$ are groups for $n \geq 1$.*

Proof. First, we note that the homotopy is well defined, as it fixes x_0 for any choice of $f \in [f]$. Using the previous remark, we conclude the proof by applying the argument from **Theorem 1.6** in one co-ordinate. $\square$

It turns out that $\pi_n(X, x_0)$ is abelian for $n \geq 2$, which is not necessarily the case for the fundamental group of X. E. Čech proposed this in a 1932 paper, which was rejected for the Zurich ICM since $\pi_n(X)$ do not generalise $\pi_1(X)$ as originally desired.[11]

The homotopy creates enough space. Given two spheres f, g, we can shrink (if necessary), re-order and then concatenate as shown below.



To formalize this notion one can apply the **Eckmann-Hilton Argument**. From now on we adopt the + notation to emphasize the abelian concatenation property on $\pi_n(X, x_0)$.

With this powerful construction at hand, it is too tempting to not ask the following question: given $\pi_n(X, x_0)$, what type of group is it (isomorphic to)?

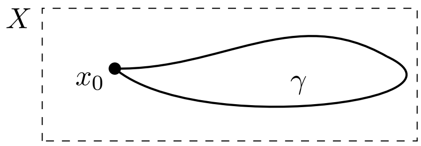
In the following part of this discussion, we attempt to answer this question by computing the higher homotopy groups of spheres. We warn the reader, however, not to get too excited. Computing $\pi_n(S^k)$ has historically proven to be a non-trivial process, and we shall provide a glimpse into this reality.

3 A Glimpse into $\pi_n(S^k)$

Here, we build up some intermediate results. The notion of homemorphisms of topological spaces is too strong for our purposes, so we weaken it by means of a homotopy.

Definition 3.1. (Homotopy of Spaces) Two spaces X, Y are homotopy equivalent if there exists maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. A space is **contractible** if it is homotopy equivalent to a point.

Example 3.1. For $X \subseteq \mathbb{R}^n$ convex with base point x_0, we have a trivial fundamental group. We write $\pi_1(X, x_0) = 0$, with 0 denoting the trivial group in one element.



We see that there is only one homotopy class, with $\gamma \simeq \mathrm{id}_{x_0}$ for all loops $\gamma \in X$. This is also true in full generality, demonstrated by the following lemma.

Lemma 3.2. *If a space (X, x_0) is contractible, then its homotopy groups are trivial.*

Proof. A contractible space is homotopic to a point, therefore there exists only the homotopy class of the constant loop $id_{x_0} : (I^n, \partial I^n) \to (X, x_0) \implies^{3.3} \pi_n(X, x_0) = 0$. □

Example 3.2. $\mathbb{R}^n$ is contractible to the origin. This also follows from the fact that it is path connected. A good visual image is to think of $\mathbb{R}^2$ as a paper contracted to a point.

Next, we observe that given a map $\varphi : (X,_x 0) \to (Y, y_0)$, there is an induced map

$$\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0) \quad \text{with} \quad \varphi_*[f] = [\varphi \circ f]$$

that is a homomorphism. One simply checks that $\varphi_*[f * g] = \varphi_*[f] * \varphi_*[g]$ where we remind the reader that $*$ is the concatenation operation. To build up on this, we have the following result on homotopy equivalent spaces and their fundamental groups.

Corollary 3.3. $(X, x_0) \simeq (Y, y_0) \implies \pi_n(X, x_0) \cong \pi_n(Y, y_0)$

Proof. Consider the induced maps on homotopy groups, with homotopy $\varphi \circ \psi = id_Y$

$$\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0) \quad ; \quad \psi_* : \pi_n(Y, y_0) \to \pi_n(X, x_0)$$

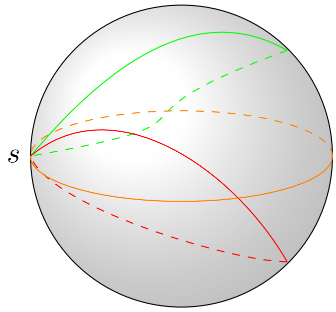
then $(\varphi_* \circ \psi_*)[f] = [\varphi \circ (\psi f)] = [(\varphi \circ \psi) f] =^{(\varphi \circ \psi = \mathrm{id})} [f]$ by homotopy equivalence. □

Here we say π is a functor from the category of topological spaces $\mathbf{Top}_*$ with base points, to **Grp**, the category of groups; a brief remark for our category theorists.

Next is our first non-trivial computation.

Lemma 3.4. $\pi_1(S^n) = 0$ *for* $n \geq 2$.

Visual Proof. For $n = 2$, we fit loops into S^2. We clearly see that all loops based at s are homotopic, i.e. there exists only the trivial homotopy class.



Therefore we can always contract all loops to the identity. For $n > 2$ one may need to be more rigorous, but the idea remains fundamentally the same. □

Remark. If X is path connected and has a trivial fundamental group, we say X is **simply connected.**

This is enough build-up, meaning we are ready to tackle the main ideas of this paper.

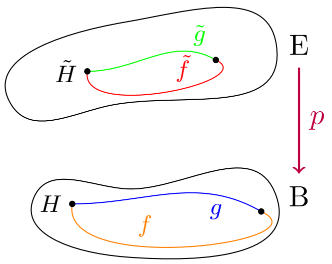
4 Fibrations & The Lifting Problem

We have seen how difficult computing homotopy groups of spheres is. If we want to do this for a more general class of topological spaces, we need to develop some more tools. This is precisely what Serre did in a 1951 paper [7], where he developed the Serre fibration specifically for the purpose of computing homotopy groups.

We now discuss the lifting problem. Let us start with the following setup

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed] \arrow[r, "f"] & B \end{tikzcd}$$

with p a continuous fixed map, and f continuous. We would like to know when the map f lifts through p to another map in E. Here is how one could visualize this.



As seen, p^{-1} ”lifts” a homotopy H to a homotopy $\tilde{H}$. This is precisely what we look for.

Definition 4.1 (Homotopy Lifting Property)**.** A map $p : E \to B$ is said to have a homotopy lifting property with respect to a topological space X if

1. for all maps $f : X \to E$
2. for all homotopies $G : X \times [0,1] \to B$ of the map $p \circ f$

there extsist a homotopy

$$F : X \times [0,1] \to E$$

such that

1. $F_0 = f$ *the homotopy F starts with f*
2. $p \circ F = G.$ *the homotopy on E projects to the homotopy on B*

A commutative diagram is a bit nicer to work with.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"'] \arrow[ur, dashed, "F"] & B \end{tikzcd}$$

with i_0 given by $x \mapsto (x, 0)$. We note that it is way too terse to ask that the maps strictly satisfy the diagram. For all practical purposes, we can ask that the lift exists up to homotopy.

Below is another useful construct for our next proof, a generalisation of the HLP.

Definition 4.2 (Relative Homotopy Lifting Property)**.** Let

$$T := (X \times \{0\}) \cup (A \times [0, 1]).$$

We say p has the homotopy lifting property for a pair (X, A) if

1. for any homotopy $F : X \times [0, 1] \to B$
2. for any lifting $\tilde{f} : T \to E$ of $f = F\big|_T$

there exists a homotopy $G : X \times [0, 1] \to E$ such that $p \circ G = F$ and $\tilde{F}\big|_T = \tilde{g}$.

Remark. Notice that $A = \phi \implies T = X \times \{0\}$ gives the standard definition of HLT.

In simple words, a homotopy on B corresponds to a homotopy on E, with an extension of the initial lift by using $\tilde{f}$. We will use this multiple times in the proof.

We define both notions of fibrations for the sake of completeness.

Definition 4.3 (Hurewicz Fibration)**.** A Hurewicz fibration is a surjection $p : E \to B$ that satisfies the homotopy lifting property with respect to all maps of the form

$$i : A \times \{0\} \hookrightarrow A \times I$$

Definition 4.4 (Serre Fibration)**.** A Serre fibration is a surjection $p : E \to B$ that satisfies the homotopy lifting property with respect to all maps of the form

$$i : I^n \times \{0\} \hookrightarrow I^n \times I \quad \text{for} \quad n \geq 0.$$

Remark. Every Serre fibration is a Hurewicz fibration.

Definition 4.5 (Fiber)**.** If $p : E \to B$ is a map of spaces and $b \in B$, then $p^{-1}(b) \subset E$ is the fiber of p over b.

Definition 4.6 (Subspace Boundary)**.** We define

$$J^n = (I^n \times \{0\}) \cup (\partial I \times I) \subset \partial I^{n+1} \subset I^n$$

to be the complement of I^{n-1}, the front face of I^n. We use the convention $J^0 = 0$.

Theorem 4.7. *Let p be a Serre fibration with fiber F. Assume further that B is path-connected. Then, the fibration*

$$(F, e_0) \hookrightarrow (E, e_0) \to^p (B, b_0)$$

induces a sequence of the form

$$\cdots \to \pi_{n+1} \to^\delta \pi_n(F, e_0) \to^{i_*} \pi_n(E, e_0) \to^{p_*} \pi_n(B, b_0) \to^\delta \dots$$
$$\cdots \to^\delta \pi_0(F, e_0) \to^{i_*} \pi_0(E, e_0) \to^{p_*} \pi_0(B, b_0).$$

Proof. The main idea here is to construct the map δ, which we like to call the connecting homomorphism. We let

$$(B, b_0) \ni [\alpha] \ni \alpha : (I^n, \partial I^n) \to (B, b_0)$$

and define $c_{e_0} : J^{n-1} \to E$ be the constant map at e_0. Then, we find that

$$\begin{tikzcd} J^{n-1} \arrow[r, "c_{e_0}"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "\alpha"'] \arrow[ur, dashed, "\beta"] & B \end{tikzcd}$$

commutes. Therefore the homotopy lifting property yields β as shown above such that $p \circ \beta = \alpha$ and $\beta(J^{n-1}) = e_0$. Next, we define $\delta[\alpha] \in \pi_{n-1}(F, e_0)$ to be the element represented by

$$\beta(-, 1) : I^{n-1} \to F, \ t \mapsto \beta(t, 1).$$

Indeed, we note the following.

1. The boundary of $I^{n-1} \times \{1\}$ is a subset of J^{n-1} and is thus mapped to e_0.
2. The image of $I^{n-1} \times \{1\}$ is contained in $(p^{-1} \circ \alpha)(I^{n-1} \times \{1\}) = p^{-1}(b_0) = F$. This follows from the fact that $I^{n-1} \times \{1\} \subset \partial I^n$ and α maps ∂I^n by definition to b_0.

It remains to show that δ is well defined on homotopy classes, i.e. is independent of the choice of $[\alpha]$. This part of the proof gets a bit technical, so we omit the details. □

Corollary 4.8. *This sequence is exact. For the sets π_0, exactness is understood in the sense that ker(p_*) = im (i_*), but without homomorphism as π_0 are not groups.*

Proof. It is indeed good practice to continually refer to the diagram below.

$$\begin{tikzcd} J^n \arrow[r, "k"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "l"] & B \end{tikzcd}$$

1. Exactness at $\pi_n(E, e_0)$. Consider the composition

$$p \circ i \ : F \to B$$

where i is the inclusion map $F \hookrightarrow E$. The map p sends the fiber F to b_0, therefore the induced composition $p_* \circ i_* = 0$ implies that $\mathrm{im}(i_*) \subset \ker\,(p_*)$. For the other inclusion,

1. choose $\pi_n(E, e_0) \ni [\alpha] \ni \alpha : I^n \to E$ such that $p_*[\alpha] = [p \circ \alpha] = 0$
2. Assume there exists $h : I^n \times I \to B$ that is a homotopy relative to ∂I^n from $p \circ \alpha$ to to the constant map c_{b_0}.
3. Define $k : J^n \to E$ by $k\big|_{I^n \times \{0\}} = \alpha$ and constant e_0 on the other faces.

$$\begin{tikzcd}
J^n \arrow[r, "k"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "l"'] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

Then the homotopy lifting property guarantees the existence of l. Define $l' := l\big|_{I^n \times \{1\}}$, and note that

$$p \circ l'(s, 1) = p \circ l(s, 1) = h(s, 1) = b_0$$

follows as h is a homotopy between $p \circ \alpha$ and c_{b_0}. Therefore $\mathrm{im}(l') \subset p^{-1}(b_0) = F$ and $l'(\partial I^n, 1) = e_0$. This is enough to conclude that $[l'] \in \pi_n(F, e_0)$ with $i_*[l'] = [i \circ l'] = [\alpha]$ by the homotopy l. The exactness is therefore established with $\ker(p_*) \subset \mathrm{im}(i_*)$

2. Exactness at $\pi_n(B, b_0)$. Let $\pi_n(B, b_0) \ni [\beta] \ni \beta : I^n \to E$. Then, for $\alpha = p \circ \beta$ we can take the same β as the lift in **Theorem 4.7**. Then we find that

$$\delta \circ p_*[\beta] = [\beta(-, 1)].$$

Note, however, that $I^{n-1} \times \{1\} \subset \partial I^n \subset \partial I^n$ implies that $\beta(-, 1)$ is the constant map c_{e_0}. Hence, the inclusion $\mathrm{im}(p_*) \subset \ker(\delta)$ from $\delta \circ p_* = 0$. Next, suppose

$$\pi_n(B, b_0) \ni [\alpha] \ni \alpha : I^n \to B.$$

Then for β in

$$\begin{tikzcd}
J^{n-1} \arrow[r, "c_{e_0}"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
I^n \arrow[r, "\beta"'] \arrow[ur, dashed, "\alpha"] & B
\end{tikzcd}$$

we get $\beta(-,1) \sim c_{e_0}$ relative to ∂I^{n-1}. Finally, we define $\gamma : I^{n-1} \times I \to E$ such that

$$\gamma(s,t) = \begin{cases}\beta(s,2t) & 0 \leq t \leq \frac{1}{2} h(s,2t-1) \quad \frac{1}{2} \leq t \leq 1\end{cases}$$

Then we indeed have that $[\gamma] \in \pi_n(E, e_0)$. The composition $p \circ h$ is constant, therefore $p \circ \gamma \simeq p \circ \beta = \gamma$. The inclusion $\ker(\delta) \subset \mathrm{im}(p_*)$ is induced by $[\alpha] = p_*[\gamma]$ and therefore concludes the exactness argument.

3. Exactness at $\pi_{n-1}(F, e_0)$. Let $\pi_n(B, b_0) \ni [\beta] \ni \beta \ : I^n \to E$. Then, we see that β as in **Theorem 4.7** shows that $c_{e_0} \simeq \beta(-,0) \sim \beta(-,1)$ relative to E. Therefore the inclusion $\mathrm{im}(\delta) \subset \ker(i_*)$ follows from $i_* \circ \delta[\alpha] = 0$. Next, let

1. $\pi_{n-1}(F, e_0) \supset \ker(i_*) \ni \gamma : I^{n-1} \to F$ be in the kernel, then $i_*[\gamma] = 0$.
2. $h : I^{n-1} \times I \to E$ be a homotopy between $i \circ \gamma$ and c_{e_0} relative to ∂I^{n-1}.

Notice $\alpha = p \circ h$ is an element of $\pi_n(B, b_0)$, and we can choose $\beta = h$ to be the diagonal homotopy lift. The inclusion $\ker(i_*) \subset \mathrm{im}(\delta)$ follows from the fact that $\delta[\alpha] = [\gamma]$.

This extensive argument is indeed sufficient to conclude the proof. □

This corollary is a key transition point in our discussion, and is the foundation of all the main results in this paper. An immediate consequence is the following,

Corollary 4.9. *For $n \geq 2$, the fibration*

$$\mathbb{Z} \hookrightarrow \mathbb{R} \to S^1$$

yields $\pi_n(S^1) = 0$.

Proof. The induced long exact sequence of homotopy groups is given by

$$\cdots \to \pi_n(\mathbb{Z}) \to \pi_n(\mathbb{R}) \to \pi_n(S^1) \to \pi_{n-1}(\mathbb{Z}) \ldots$$

We immediately get from **(Example 3.1)** that $\pi_n(\mathbb{Z}) = \pi_n(\mathbb{R}) = 0$ for $n > 0$. Then, the exactness of the sequence implies the triviality of $\pi_n(S^1)$ for $n \geq 2$. □

Remark. Here we present a somewhat unconventional proof, contrary to a standard proof using covering space theory. We specifically make this choice to emphasize the full strength of the resulting long exact sequence.

5 Applications to Homotopy Groups

Definition 5.1 (The Hopf Fibration)**.** Define

$$S^3 := \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$$

to be the 3-sphere in the two-dimensional complex plane $\mathbb{C}^2$. Next, let

$$\eta(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2^2|)$$

be the Hopf fibration, with $\bar{z}_2$ denoting the complex conjugate of z_2.

Corollary 5.2. *η is well-defined.*

Proof.
$$\left|2z_1\bar{z}_2\right|^2 + \left(|z_1|^2 - |z_2|^2\right)^2 = 1 \implies im(\eta) \subseteq S^2$$
□

Theorem 5.3. *$\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$*

Proof. Notice that the fibration given by

$$S^1 \hookrightarrow S^3 \to_\eta S^2$$

induces a long exact sequence of homotopy groups by **Corollary 4.8**,

$$\cdots \to \pi_n(S^1) \to \pi_n(S^3) \to \pi_n(S^2) \to \pi_{n-1}(S^1) \to \ldots \quad .$$

We know by **Corollary 4.9** that $\pi_n(S^1)$ is trivial for all $n \geq 2$, hence

$$\ldots \boxed{\to 0 \to \pi_n(S^3) \to \pi_n(S^2) \to 0} \to \ldots$$

for which applying **Lemma 0.6** establishes the claim. □

Remark. We have the short exact sequence shown above for $\pi_{n-1}(S^1) = 0$, which explains the choice of $n \geq 3$.

Definition 5.4. Let X be a topological space. For a base point $b \in X$, define

$$P(X, b) := \{\gamma : I \to X \mid \gamma(0) = b\}$$

to be the path space of paths in X starting at b.

Remark. A loop is a special path where $\gamma(1) = \gamma(0)$. Here, this is not always the case.

Definition 5.5. For a topological space X, we define

$$\Omega_b X := \{\gamma : I \to X \mid \gamma(0) = \gamma(1) = b\}$$

to be the space of all loops based at $b \in X$.

We have already extensively discussed loops in space, so this concept is not alien to us.

Definition 5.6 (The Path-loop Fibration.)**.** The map $p : P(X, b) \to X$ defines

$$\Omega_b X \hookrightarrow P(X, b) \to^p (X, b)$$

a Serre fibration with fiber $\Omega_b X$.

Lemma 5.7. *$P(X, b)$ is contractible for any $X,\ b \in X$.*

Proof. We show that for maps

$$\begin{aligned} f : P(X, b) \to b \quad & \gamma \mapsto \gamma(0) \\ g : b \to P(X, b) \quad & b \mapsto \gamma_b \end{aligned}$$

with γ_b the constant path on b, that *(see Definition 3.1)*

$$f \circ g \simeq id_b \quad \& \quad g \circ f \simeq id_{P(X,b)}.$$

If we can retract $P(X, b)$ to a point, then we are done. Clearly, $f \circ g$ yields

$$b \mapsto_g \gamma \mapsto_f \gamma(0) := b = id_b$$

the identity on b. Next, notice that $g \circ f$ we always have the constant path based at b.

$$P(X, b) \mapsto_f b \mapsto_g \gamma_b \in P(X, b)$$

We define

$$h(s, t) = \gamma_b(st)$$

such that

1. $h(0, t) = id_b$ *the homotopy fixes the identity*
2. $h(1, t) = \gamma_b(t)$ *as well as the constant map*
3. $h(s, 0) = \gamma_b(0) = b$ *the first startpoint is fixed*
4. $h(s, 1) = \gamma_b(s) = b$ *the second endpoint is fixed*

therefore, a homotopy $\gamma_b \simeq id_{P(X,b)}$ for every constant path γ_b. The proof is concluded with $g \circ f \simeq id_{P(X,b)}$. □

Theorem 5.8. $\pi_n(\Omega X) \cong \pi_{n+1}(X)$

Proof. From the fibration

$$\Omega_b X \hookrightarrow P(X, b) \to X_b$$

we have the following long exact sequence (**Corollary 4.8**).

$$\cdots \to \pi_{n+1}\big(P(X, b)\big) \to \pi_{n+1}(X) \to \pi_n(\Omega X) \to \pi_n\big(P(X, b)\big) \to \dots \quad .$$

Since $P_b X$ is contractible (**Lemma 5.7**), we immediately apply **Lemma 3.2** to get

$$\cdots \to 0 \to \pi_{n+1}(X) \to \pi_n(\Omega X) \to 0 \to \dots \quad .$$

Lemma 0.6 concludes the proof. □

Acknowledgements

This paper was prepared for a course in Algebraic Topology taught by Professor Alan Huckleberry and Mr Dmytro Rudenko; for whom I am immensely grateful for their great insight, help and support.

References

[1] Allen Hatcher. *Algebraic Topology*.

[2] Witold Hurewicz and Norman E. Steenrod. Homotopy relations in fibre spaces. *Proc. Nat. Acad. Sci. U. S. A.*, 27:60–64, 1941.

[3] Michael Hutchings. Introduction to higher homotopy groups and obstruction theory.

[4] Sanatil Jana. The hopf fibration.

[5] William M. Landau. *The Fundamental Group*. Expository.

[6] Ieke Moerdijk and Javier J. Gutiérrez. *Course on Algebraic Topology (Fall 2014)*.

[7] Jean-Pierre Serre. Homologie singulière des espaces fibrés. applications. *Ann. of Math.*, 54, 1951.

[8] Derek Sorensen. *An Introduction to Fiber Bundles and Fibrations*.

[9] Edwin H. Spanier. *Algebraic Topology*. Springer-Verlag, 1981.

[10] Robert M. Switzer. *Algebraic Topology - Homotopy and Homology*. Springer-Verlag, 1975.

[11] Eduard Čech. Höherdimensionale homotopiegruppen. *Verhandlungen des Internationalen Mathematikerkongress*, 1932. Zürich.