The Banach-Tarski Paradox

An Exposition

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Overview

What is a Free Group?

Define

- S a set
- $S^{-1} = \{s \in S \ : s^{-1}\}$ such that $s_i s_i^{-1} := e$
- $T = S \cup S^{-1}$
- $\langle S \rangle := \{ \ \boldsymbol{t_1 t_2} \ \ldots \ \boldsymbol{t_n} \ : \ t_i \ t_{i+1} \neq e \ , \ t_i \in T, \ n \in \mathbb{N}_0\}$

where $w_n \in \langle S \rangle$ is a reduced word of length n, and $t_0 := e$.

We denote the free group with $F_S := \langle S \rangle$ of rank $|S|$.

Free F_2

Similarly,

- $S = \{\sigma, \tau\}$ with $|S| = 2$.
- $S^{-1} = \{\sigma^{-1}, \tau^{-1}\}$
- $T = \{\sigma, \sigma^{-1}, \tau, \tau^{-1}\}$

and $F_2 = \langle \sigma, \tau \rangle$. How does F_2 look like?

- $\tau\tau\sigma^{-1}\sigma^{-1}\cdots \in F_2$
- $\tau\sigma^{-1}\sigma\tau^{-1} \in F_2$?

How does F_2 look like?

Define $W(t) := \{w \in \langle \sigma, \tau \rangle : w_1 = t\}$ for $t \in T$. Recall,

$$T = \{\sigma, \tau, \tau^{-1}, \sigma^{-1}\}$$

- $W(\sigma)$
- $W(\sigma^{-1})$
- $W(\tau)$
- $W(\tau^{-1})$

Then, $F_2 = \{e\} \cup W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1})$

A Paradoxical Decomposition

Weird, right?

Apply σ^{-1} to $W(\sigma)$, then

$$\sigma^{-1}W(\sigma) = W(\sigma) \cup W(\tau) \cup W(\tau^{-1}) \cup e = F_2 \setminus W(\sigma^{-1})$$
$$\tau^{-1}W(\tau) = W(\tau) \cup W(\sigma) \cup W(\sigma^{-1}) \cup e = F_2 \setminus W(\tau^{-1})$$

$$\implies \sigma^{-1}W(\sigma) \cup W(\sigma^{-1}) = F_2 = \tau^{-1}W(\tau) \cup W(\tau^{-1}).$$

Observe that

$$[\sigma^{-1}W(\sigma)] \cap [\tau^{-1}W(\tau)] \neq \{\}.$$

Going from F_2 to S_2

This is not geometric - yet. Can we fit F_2 into finite Euclidean space?

Goal: A Free group of rotations,

$$G\langle\sigma,\tau\rangle \in \mathbb{R}^3 \ : \ G\langle\sigma,\tau\rangle \cong F_2$$

acting on the unit Sphere S^2 in R^3.

- $\sigma \circ \tau \neq \tau \circ \sigma$
- e uniquely determined by w_0

Construction of $G\langle\sigma,\tau\rangle$

For simplicity, we choose rotations around the x, y axes.

$$\sigma = R_x(\theta) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\tau = R_y(\theta) := \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Exercise: Show that $\sigma \circ \tau \neq \tau \circ \sigma$.

Choice of θ

Proposition, *$G\langle\sigma,\tau\rangle \cong F_2$ is a Free Group $\implies$ θ is an irrational multiple of π*

Proof. Assume the contrary, let $\theta = 2\pi k$ for $k \in \mathbb{Q}\big|_{[0,1]}$.

Then $k = \frac{p}{q} \implies q\,\theta = 2\pi p$. We know

$$\begin{aligned} \sigma^q &:= [R_x(\theta)]^q \\ &= R_x(q\theta) \\ &= e \end{aligned}$$

But $|\sigma^q| = q \neq 0 \implies G\langle\sigma,\tau\rangle$ is not a Free Group. □

Banach-Tarski paradox

first version

Theorem 1, *There exists a countable subgroup G of $SO(3)$, and a partition*

$$G = G_1 \uplus G_2 \uplus G_3 \uplus G_4 \tag{1}$$

into disjoint sets G_1, G_2, G_3, G_4, such that one can write

$$G = G_1 \uplus \sigma\ G_2 = G_3 \uplus \tau\ G_4 \tag{2}$$

for some rotations $\sigma, \tau \in SO(3)$.

Proof. G is precisely $G\langle\sigma, \tau\rangle$. Choose

$$G_1 = W(\sigma) \cup \{e, \sigma^{-1}, \sigma^{-2}, \dots\} \quad G_2 = W(\sigma^{-1}) - \{\sigma^{-1}, \sigma^{-2}, \dots\}$$
$$G_3 = W(\tau) \quad G_4 = W(\tau^{-1}) \quad \square$$

Orbits of S^2

Define $S^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}$, the Unit Sphere in $\mathbb{R}^3$.

Lemma 2, $x \sim y \iff \exists\, \rho \in \langle \sigma, \tau \rangle\ : \rho \circ x = y$. *Show that* $\sim$ *is an equivalence relation for* $x, y \in S^2$.

Proof. Exercise.

Therefore $\sim$ partitions S^2 into equivalence classes, which are disjoint sets. We refer to equivalence classes as Orbits.

Did we miss something?

Dealing with Poles

Every axis of rotation intersects S^2 at two poles, unchanged under rotation. Why is this problematic?

$$\rho_1 \circ x = \rho_2 \circ x \implies (\rho_1^{-1}\rho_2) \circ x = x$$
$$\implies x \text{ is a pole and } \rho_1^{-1}\rho_2 \text{ is a non-trivial identity.}$$

Let D be the set of all poles. Then $|D| = 2\,|G\langle\sigma,\tau\rangle|$, and now $G\langle\sigma,\tau\rangle$ acts freely on $S^2 \; - \; D$.

The Axiom of Choice

Let us define a set E with Orbits of $S^2 - D$,

$$E = S^2 - D \ / \ G\langle \sigma, \tau \rangle = \{[x] : x \in S^2 - D\}.$$

Note that E is infinite. By the Axiom of Choice,

- Pick a point from each Orbit $[x] \in E$
- Let M be the set of all such points.

Therefore, by Banach-Tarski's 1^{st} Paradox,

$$\begin{aligned} S^2 - D &= G\langle \sigma, \tau \rangle \circ M \\ &= G_1 \circ M \uplus G_2 \circ M \uplus G_3 \circ M \uplus G_4 \circ M \\ &= G_1 \circ M \uplus \sigma \ G_2 \circ M \\ &= G_3 \circ M \uplus \tau \ G_4 \circ M \quad \square \end{aligned}$$

This result is known as *The Hausdorff Paradox.*

Dealing with D

Now we deal with D, the set of Poles.

Lemma 3, *Let D be a countable subset of S^2. Then*

$$\begin{aligned} S^2 &= \Sigma_1 \uplus \Sigma_2 && \textit{such that} \\ S^2 - D &= \Sigma_1 \uplus \varphi \circ \Sigma_2 && \textit{for some } \varphi \in G\langle \sigma, \tau \rangle. \end{aligned}$$

Proof.
Choose φ arbitrarily. We can find a rotation such that $\varphi^i \circ D \cap \varphi^j \circ D = \{\}$ for $i \neq j$, and we make a clever choice of

$$\Sigma_1 = S^2 - \Sigma_2 \text{ and } \Sigma_2 = D \cup \varphi \circ D \cup \varphi^2 \circ D \cup \ldots$$

and we are done.

Partition of S^2

At Last!

Combining *The Hausdorff Paradox* with **Lemma 3**, we get that

$$S^2 = \Gamma_1 \uplus \cdots \uplus \Gamma_8.$$

Further,

$$S^2 = \coprod_{i=1}^{4} R_i \circ \Gamma_i = \coprod_{i=5}^{8} R_i \circ \Gamma_i.$$

S^2 to B^3

How?

Recall $S^2 = \{s \in \mathbb{R}^3 \ : ||s|| = 1\}$ and $B^3 := \{b \in \mathbb{R}^3 \ : ||s|| \leq 1\}$.

The punctured ball $B^3 - \{0\}$ can be thought of as the product of the sphere S^2 and the interval $(0, 1]$.

$f : S^2 \times (0, 1] \to B^3 - \{0\}$ such that

$$f(s, r) = r \cdot s \text{ for } x \in S^2, \ r \in (0, 1].$$

Corollary, (Puncture at the Origin)
$B^3 - \{0\}$ *is equi-decomposable with* B^3.

Proof. Exercise.

Hint: Use a similar argument (trick) to **Lemma 3**.

Axiom of Choice and Criticism

A Discussion

- Is it really a paradox?
- Is not really AC's fault - 1^{st} Version
- $\implies$ Infinity is weird.
- Subsets have no measure (Non-Lebesgue Measurable)
- Mathematics would fall without Axiom of Choice

- Outlook:
  - Mathematically ideal, infintely complex partitions
  - Quaternions can collide at high energies and turn into more particles

$\implies$ AC Enjoyer

References

[1] Terence Tao. The Banach-Tarski Paradox.

[2] Avery Robinson. The Banach-Tarski Paradox.

[3] Teun van Wesel. Non-measurable Sets and the Banach-Tarski Paradox.

Questions?

- **Exercises,**
  - Slides $12, 17, 26$
  - $oelshinawy@constructor.university$
  - Feel free to reach out :)